More on a Binary Encoding
Edge Coloring Formulation

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All-different constraint:

- Set $M$ of objects $M = \{M_1, \ldots, M_m\}$

- Set $S$ of subsets of $M$, $S = \{S_1, \ldots, S_s\}$

Problem:

- Assign colors from $\{0, \ldots, K - 1\}$ to each $M_j \in M$ s.t. all colors assigned to $M_j \in S_i$ are distinct $\forall i$

- Either $K$ is given (feasibility problem) or minimize $K$
Examples:

- Graph coloring (edge or vertex)
- Scheduling on identical machines
- Classification
- Clustering
General ILP Formulation

Variables:

\[ z_j = \text{color assigned to } j \]

Constraints:

\[ \text{???} \]

\[ \frac{1}{2}(0, 1, 2, 3) + \frac{1}{2}(2, 3, 0, 1) = (1, 2, 1, 2) \]

Permutahedron: [Balas, 75]
Weakly k-majorized vectors: [Dahl, M., 96]
0/1 ILP Formulations

Unary representation

Variables:

\[ z_{jk} = \begin{cases} 
1 & \text{if } j \text{ gets color } k \\
0 & \text{otherwise} 
\end{cases} \]

Constraints:

\[ \sum_k z_{jk} = 1 \quad \forall j \quad (j \text{ gets one color}) \]

\[ \sum_{j \in S_i} z_{jk} \leq 1 \quad \forall i, \forall k \quad (\#\text{color } k \text{ in } S_i) \]

- #variables: \( Km \)

- #constraints: \( m + sK \) (+ bounds inequalities)
Column Generation

Edge Coloring: Covering with matchings

Variables: For each $T$ inducing a matching

$$z_T = \begin{cases} 1 & \text{if } T \text{ used in cover} \\ 0 & \text{otherwise} \end{cases}$$

Constraints:

$$\sum_{T \mid e \in T} z_T \geq 1 \quad \forall e \ (e \text{ covered})$$

- #variables: #matchings
- #constraints: $m$ (± bounds ineqs)

[Nemhauser, Park, 91]
[Lee, Leung, 93]

Vertex Coloring: [Mehrotra, Trick, 95]
Binary representation

Variables:

\[ n := \lceil \log K \rceil \]

To each object \( j \) associate an \( n \)-vector
\[ x_j := (x_{j1}, \ldots, x_{jn}). \]

\( x_j \) valid: \( x_j \) is the binary representation of \( k \in \{0, \ldots, K - 1\} \).

\( k \) has representation \( C_k, \forall k \in \{0, \ldots, K - 1\} \)
\[ Q(m, n, K) = \{ X \in \{0, 1\}^{m \times n} \mid x_j \text{ valid } \forall j, \]

all rows different\}

Example:

\[
\begin{align*}
C_4 & \rightarrow \begin{pmatrix} 100 \\ 000 \end{pmatrix} \\
C_0 & \rightarrow \begin{pmatrix} 011 \\ 001 \end{pmatrix} \in Q(4, 3, 5)
\end{align*}
\]
Constraints:

- Convex hull inequalities (CHI)
- General block inequalities (GBI)
- LP-based cuts (LPC)

- #variables: $m \lfloor \log K \rfloor$

- #constraints: exponential (?),
  (separation algorithm: poly$(m, K)$)
Convex Hull Inequalities

\[ x_j \in CH := \text{conv } \{C_k \mid k = 0, \ldots, K - 1\} \]

Separation of \( x_j \):

- **Brute Force**: Compute CH at the beginning

- **LP**: \( \max \bar{x}_j \pi_j - \sigma \)
  \[ \text{s.t. } C_k \pi_j - \sigma \leq 0 \quad \forall k \in \{0, \ldots, K\} \]
  \[ -1 \leq \pi_j \leq 1 \]

If the optimal value is positive, \( \pi x_j \leq \sigma \) is a separating hyperplane
**General Block Inequalities** [Lee, 02]

For $0 \leq d \leq n$, $B$ any 0/1 $n$-vector:

$$h(B, d, n) = |\{C_k \mid \text{Ham}(B, C_k) = d\}|$$

Example:

$K = 5, n = 3, B = (100)$

$$C = \begin{pmatrix} 000 & \leftarrow C_0 \\ 001 & \leftarrow C_1 \\ 010 & \leftarrow C_2 \\ 011 & \leftarrow C_3 \\ 100 & \leftarrow C_4 \end{pmatrix}$$

$$h(B, 0, n) = 1$$
$$h(B, 1, n) = 1$$
$$h(B, 2, n) = 2$$
$$h(B, 3, n) = 1$$
If 4 distinct rows, then the sum of the Hamming distance of these rows to $B$ is at least:

$$0 \cdot h(B, 0, n) + 1 \cdot h(B, 1, n) + 2 \cdot h(B, 2, n) = 5$$

i.e.

$$\sum_{j=1}^{4} ((1 - x_{j1}) + x_{j2} + x_{j3}) \geq 5$$

**Theorem:** [Lee, 02] If $K = 2^n$, almost all GBI are facet of $Q(m, n, K)$.

**Theorem:** Separation of $X$ from $Q(m, n, K)$ with GBI can be done in time polynomial in $m$ and $n$. 
For $1 \leq p \leq 2^n$, $p$ can be written uniquely as

$$p = h + \sum_{k=0}^{t} \binom{n}{k}, \text{ with } 0 \leq h < \binom{n}{t+1}.$$ 

$t$: $n$-binomial size of $p$

$h$: $n$-binomial remainder of $p$

$(S, S')$ ordered partition of $N$

$x \in \mathbb{R}^n$

$$x(S, S') := \sum_{i \in S} x^i + \sum_{i \in S'} (1 - x^i).$$

A $t$-light partition for $x \in \mathbb{R}^n$ is a partition $(S, S')$ with $x(S, S') < t$.

**Lemma 1** Let $p$ satisfy $1 \leq p \leq 2^n$, and let $t$ be the $n$-binomial size of $p$. Then for $x \in \mathbb{R}^n$ with $0 \leq x \leq 1$, at most $4(n+1)^2p^2 + (n+1)p$ partitions are $(t+1)$-light for $x$. 


Separation Algorithm for GBI

(0) Let \( X \in [0, 1]^{m \times n} \), and let \( t \) be the \( n \)-binomial size of \( m \).

(1) For each \( e \in M \), compute the set \( T_e \) of all \((t + 1)\)-light partitions for \( x_e \).

(2) Then, for each partition \((S, S')\) in \( \cup_{e \in M} T_e \):

   (2.a) Compute \( F \subseteq M \) such that, for each \( e \in F \), \((S, S')\) is a \((t + 1)\)-light partition for \( x_e \).

   (2.b) Order \( F = \{e_1, \ldots, e_f\} \) such that

   \[
   x_{e_i}(S, S') \leq x_{e_{i+1}}(S, S') \quad \text{for} \quad i = 1, \ldots, f - 1.
   \]

   (2.c) If one of the partial sums \( \sum_{i=1}^{k} x_{e_i}(S, S') \), for \( k = 2, \ldots, f \) is smaller than \( \kappa(k, n) \), then

   \[
   L := \{e_1, \ldots, e_k\} \quad \text{and} \quad (S, S') \quad \text{generate a violated GBI for} \quad X.
   \]

(1) can be done in polynomial time

(Reverse Search [Avis, Fukuda, 96])
LP-cuts

**Theorem 2:** \( \bar{X} \in [0, 1]^{m \times n} \)

\( \exists \) efficient algorithm that checks whether \( \bar{X} \in Q(m, n, K) \), and if not, determines a hyperplane separating \( \bar{X} \) from \( Q(m, n, K) \).

\( \pi \in \mathbb{R}^{m \times n} : \) coefficients

\( \sigma \): right hand side

Kronecker product:

\[
(\bar{X} \ast \pi) := \sum_{i,j} \bar{X}_{ij} \pi_{ij}
\]

\[
\max \; \bar{X} \ast \pi - \sigma
\]
\[
\text{s.t.} \; X \ast \pi - \sigma \leq 0 \quad \forall X \in Q(m, n, K)
\]
\[
-1 \leq \pi \leq 1
\]

If the optimal value is positive, \( \pi \ast X \leq \sigma \) is a separating hyperplane

Maximum violation for coefficients in \([-1, 1]\)
Optimizing over $Q(m, n, K)$ is just optimal assignment of rows to available colors. Given $\pi = \begin{pmatrix} 0.5 & 1 \\ 0.2 & 0.8 \end{pmatrix}$:

$$\max X \ast \pi$$

s.t. $X \ast \pi - \sigma \leq 0 \quad \forall X \in Q(2, 2, 3)$

If row $j$ of $X$ is chosen as $C_k$, it will cost $C_k \pi_j$
Assignment formulation:
(optimize $X \ast \pi$ for $X \in Q(m, n, K)$):

$$z_{jk} = \begin{cases} 
1 & \text{if row } j \text{ assigned to color } k \\
0 & \text{otherwise}
\end{cases}$$

Primal: (P)

$$\max \sum_{j,k} (\pi_j \ C_k \ z_{jk})$$

s.t. $\sum_k z_{jk} = 1 \quad \forall j \quad (\alpha_j)$

$$\sum_j z_{jk} \leq 1 \quad \forall k \quad (\beta_j)$$

$Z \geq 0$

Dual: (D)

$$\min \sum_j \alpha_j + \sum_k \beta_k$$

s.t. $\alpha_j + \beta_k \geq \pi_j \ C_k \quad \forall j, k$

$\beta \geq 0$
Separation of $\bar{X}$:

Variables: $\pi \in \mathbb{R}^{m \times n}$, $\sigma \in \mathbb{R}$, $\alpha \in \mathbb{R}^m$, $\beta \in \mathbb{R}^K$

$$\begin{align*}
\text{max} & \quad \bar{X} \ast \pi - \sigma \\
\text{s.t.} & \quad -1 \leq \pi \leq 1 \\
& \quad \sum_j \alpha_j + \sum_k \beta_k \leq \sigma \\
& \quad \alpha_j + \beta_k \geq \pi_j C_k \quad \forall j, k \\
& \quad \beta \geq 0
\end{align*}$$

\begin{equation}
\left\{ \begin{array}{l}
\text{Opt of (D)} \leq \sigma \\
\text{positive optimal value}
\end{array} \right. 
\end{equation}

$\pi \ast X \leq \sigma$ is a separating hyperplane with maximum violation for coefficients in $[-1, 1]$

[Martin, 91]
Edge Coloring

$n$-bit edge coloring polytope of graph $G$:

$$Q_n(G) := \text{conv} \{ X \in \{0, 1\}^{E(G) \times n} :$$
$$x_e \neq x_f, \ \forall \text{ distinct } e, f \in \delta(v),$$
$$\forall v \in V(G) \}$$

Matching inequalities:

$$E' \subseteq E(G)$$
$$F \subseteq E' : \text{ maximum matching in } E'$$
$$(S, S') : \text{ partition of } \{0, \ldots, K - 1\}$$

matching inequality (MI):

$$x_{E'}(S, S') \geq |E' \setminus F|$$
Proposition 2: Let $G'$ be the graph induced by $E'$. The MI induced by $E'$ is dominated in the following cases:

(i) $G'$ is not connected  
(ii) $G'$ has a vertex $v$ saturated by every maximum matching in $G'$
(iii) $G'$ has a cut vertex $v$
(iv) $G'$ is bipartite
(0) Let $\bar{X}$ be a point in $[0, 1]^{E(G) \times N}$.

(1) For each partition $(S, S')$ of $N$:

(1.a) Compute the edges $T$ for which $(S, S')$ is a 1-light partition.

(1.b) For each non-bipartite block of the graph $G'$ induced by $T$:

(1.b.i) Compute a maximum matching $F(G')$ in $G'$.

(1.b.ii) Check if $x_{E(G')}(S, S') \geq |E(G') \setminus F(G')|$ is a violated matching inequality.
Switched walk inequalities

subpartition: \((S_i, S'_i)\) such that
\[|S_i| + |S'_i| \geq n - 1\]

\((S_1, S'_1)\) : subpartition
\((S_2, S'_2)\) : subpartition obtained by:

1. adding the only element not in \(S_1 \cup S'_1\) (if any) either to \(S_1\) or to \(S'_1\); call the resulting partition \((P_2, P'_2)\)
2. removing at most one element from \(P_2\) or at most one element from \(P'_2\).

Then \((S_2, S'_2)\) is a switch of \((S_1, S'_1)\).

Example: \(j \in S\) denoted by '+',
\(j \in S'\) denoted by '−':

\[
\begin{array}{ccc}
+ & - & - \\
+ & 0 & - \\
0 & + & - \\
0 & + & - \\
+ & + & - \\
\end{array}
\]
\{e_1, \ldots, e_k\} \text{ walk in } G
\quad (S_i, S_i') \text{ switch of } (S_{i-1}, S_{i-1}') \text{ for } i = 2, \ldots, k
\quad (S_1, S_1') \text{ and } (S_k, S_k') \text{ are the only partitions}

\text{For all } j \in S_t, \text{ if } [t+1, t'] \text{ is a maximal interval such that for all } t+1 \leq i \leq t' \text{ we have } N - (S_i \cup S_i') = \{j\}, \text{ then } j \in S_{t'+1} \text{ if and only if } t' - t \text{ is odd}

\text{For all } j \in S_t', \text{ if } [t+1, t'] \text{ is a maximal interval such that for all } t+1 \leq i \leq t' \text{ we have } N - (S_i \cup S_i') = \{j\}, \text{ then } j \in S_{t'+1} \text{ if and only if } t' - t \text{ is even.}

Then the walk and the set of subpartitions \((S_1, S_1'), \ldots, (S_k, S_k')\) form a \textit{switched walk}.

Given a switched walk, the inequality

\[(\text{SWI}) \quad \sum_{i=1}^{k} x_{e_i}(S_i, S_i') \geq 1\]

is a \textit{switched walk inequality} (SWI)
Let $N := \{0, 1, 2\}$. Consider the path of edges $(e_1, e_2, e_3, e_4, e_5)$. Associated with the sequence of edges of the path is the switched walk:

\[
\begin{align*}
e_1 & \rightarrow + \quad - \quad - \\
e_2 & \rightarrow + \quad 0 \quad - \\
e_3 & \rightarrow 0 \quad + \quad - \\
e_4 & \rightarrow 0 \quad + \quad - \\
e_5 & \rightarrow + \quad + \quad -
\end{align*}
\]

The given switched walk gives rise to the SWI:

\[
\begin{align*}
+x_0^0 & + (1 - x_1^1) \quad + (1 - x_1^2) \\
+x_0^2 & \quad + (1 - x_2^2) \\
+x_1^3 & \quad + (1 - x_3^2) \\
+x_4^1 & \quad + (1 - x_4^2) \\
+x_5^0 & \quad + x_5^1 \quad + (1 - x_5^2) \geq 1
\end{align*}
\]
Theorem 4: If $P$ is a path and $n \geq 2$, then $Q_n(P)$ is described by the SWI and the simple bound inequalities $0 \leq X \leq 1$
Odd Components inequalities:

$G$: regular graph with $|V(G)|$ even
$x$: current LP solution
$S$: edges of $G$ whose color is fixed
$G' := G - S$

For each color $C = (S, S')$:

$G'' := G' - \{\text{nodes incident to an edge whose color in } x \text{ is } C\}$

For each connected component $A$ of $G''$:
If $A$ has no perfect matching then
$A(C) : \text{edges adjacent to } A \text{ with color } C$

$x_{A(C)}(S, S') \geq 1$
**Overfull Subgraph**

$H$ induced subgraph of a simple graph $G$

- $|V(H)|$ odd
- $2|E(H)| > \Delta_G (|V(H)| - 1)$

**Overfull Subgraph Conjecture:**
[Chetwynd-Hilton 86]

If

- $\Delta_G > |V(G)|/3$
- edges of $G$ are not $\Delta_G$-colorable

Then $G$ has an overfull subgraph.

$\exists$ polynomial time alg. for finding overfull subgraphs when $\Delta_G > |V(G)|/3$ [Niessen 2000]
Test Problems (isom.):

- $k5 : K_5$ (120)
- peter: Petersen graph (120)
- of5_14_7: overfull sub., 5-reg., 14 vertices (48)
- of7_18_9: overfull sub., 7-reg., 18 vertices (13,824)
- ofsub9: overfull sub. of of7_18_9, $\Delta = 7$ (12)
- jgt18: non 4-col., 4-reg., 18 vertices [Chetwynd-Wilson 86] (2)
- jgt30: non 4-col., 4-reg., 30 vertices [chetwynd, Wilson 86] (2)
- O4_35: overfull, 4-reg., 35 vertices (2520)
- g_8_40: 8-colorable, 8-reg., 40 vertices (1)
- mered : Meredith graph, 4-reg., 70 vertices (43,646,976)
Test Problems:

- $g_{xy}$: $x$-col., $x$-reg., $y$ vertices
- peter: petersen graph
- of5_14_7: overfull sub., 5-reg., 14 vertices
- of7_18_9_5: overfull sub., 7-reg., 18 vertices
- jgt18: non 4-col., 4-reg., 18 vertices

[Chetwynd-Wilson 86]