Essays in Electronic Commerce: Game Theoretic Analysis and Optimization

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ABSTRACT

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This dissertation studies three problems motivated by electronic commerce applications.

The first problem deals with the design of revenue maximizing procurement auctions with divisible quantities in a setting where both the marginal cost and the production capacity are private information of the suppliers. We provide a closed-form solution for the revenue maximizing direct mechanism when the prior distribution of the marginal cost and the production capacity satisfies a particular regularity condition. We also present a sealed low bid implementation of the optimal direct mechanism for the special case of identical suppliers, i.e. the symmetric environment. Our results extend to other principle-agent mechanism design problems where the agents have a privately known upper bound on allocation. Examples of problems of this nature include monopoly pricing with adverse selection, forward auctions and scheduling with privately known deadlines and values.

The second problem deals with the design of the optimal sponsored search auctions used by the internet search service providers such as Google and Yahoo!. We begin with a general problem formulation which allows the privately known valuation per click to be a function of both the identity of the advertiser and the slot. We present a compact characterization of the set of all deterministic dominant strategy incentive compatible direct mechanisms for this model. This new characterization allows us to conclude that there are incentive compatible mechanisms for such an auction in a multi-dimensional type-space that *are not* affine maximizers. Next, we discuss two interesting special cases: slot independent valuation and slot independent valuation up to a privately known slot and zero thereafter. For both of these special cases, we characterize revenue maximizing and efficiency maximizing mechanisms and show that these mechanisms can be computed with a worst case computational complexity $O(n^2m^2)$ and $O(n^2m^3)$ respectively, where n is number of bidders and m is number of slots. Next, we characterize optimal rank based allocation rules and propose a new mechanism that we call the customized rank based allocation. We report the results of a numerical study that compare the revenue and efficiency of the proposed mechanisms. The results from this study suggest that customized rank based allocation rule is significantly superior to the rank-based allocation rules.

The third problem studied in this dissertation is the design and analysis of a simple online exchange for matching impatient demand with patient supply. Our proposed exchange mechanism is motivated by the limit order book mechanism used in stock markets. In this model, both buyers and sellers are elastic in the price-quantity space; however, only the sellers are assumed to be patient, i.e. only

the sellers have a price-time elasticity, while the buyers are assumed to be impatient. We define and establish the existence of the equilibrium in this model and show how to numerically compute this equilibrium. We derive a closed form for the equilibrium distribution when the demand is price independent. At this equilibrium the selling (limit order) price distribution is power tailed as is empirically observed in order driven financial markets. We extend this model to multiple competing exchanges indexed by quality.

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Notation

We denote vectors by boldface lowercase letters, e.g. $\mathbf{x} \in \mathbb{R}^N$ while matrices are denoted by boldface uppercase letters, e.g. $\mathbf{C}$. Sets are denoted calligraphic letters, e.g. $\mathcal{M}$. We denote positive and strictly positive real line by $\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively. A vector indexed by $-i$, (for example $\mathbf{x}_{-i} \in \mathbb{R}^{N-1}$) denotes the vector $\mathbf{x}$ with the i-th component excluded. We use the convention $\mathbf{x} = (x_i, \mathbf{x}_{-i})$. Scalar (resp. vector) functions are denoted by lowercase letters, e.g. $x_i(\theta_i, \theta_{-i})$ (resp. $\mathbf{x}(\theta_i, \theta_{-i})$) and conditional expectation of functions by the uppercase of the same letter, e.g. $X_i(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} x_i(\theta_i, \theta_{-i})$ (resp. $\mathbf{X}(\theta_i) = \mathbb{E}_{\theta_{-i}}[\mathbf{x}(\theta_i, \theta_{-i})]$). The possible misreport of the true parameters are represented with a hat over the same variable, e.g. $\hat{\theta}$.

Chapter 1

Introduction

Electronic commerce (e-commerce) refers to the subset of global business where negotiations and actual transactions of goods and services are conducted electronically, rather than physically. E-commerce is a general term for any type of business, or commercial transaction that involves the electronic transfer of information. This covers a range of different types of businesses from consumer-based retail sites, like Amazon.com, to auction sites, like the sponsored search auctions used by Google, and business exchanges trading goods or services between corporations and individuals.

The spectacular successes of e-commerce Multi National Companies (MNC)'s like Amazon, Google, Yahoo, eBay etc, which did not exist in the last decade, clearly indicate that E-commerce is reshaping the world economy. The electronic auction house, eBay, whose consumer auction infrastructure is facilitating consumer-to-consumer transactions, earned a revenue of $6.0 billion in the year 2006 (a 31% increase over the $4.6 billion generated in 2005). Google and Yahoo!, whose electronic - sponsored search auctions are facilitating business-to-consumer and business-to-business transactions, earned revenues of $7.14 and $5.26 billion, respectively, in 2006.

Corporations across the globe are turning to e-commerce for a number of reasons. E-commerce allows automation of many operational aspects of the business leading to a reduction in operational costs; it allows corporations to reduce transaction time and costs; in addition, data encryption to ensure that all transactions are secure. Lower communication costs in e-commerce allows corporation across the globe to trade with each other - trades which would have otherwise not been possible. See Whinston et al. [72] for an introduction to the general economic and business issues in e-commerce.

In a typical e-commerce system (or electronic market mechanism), multiple self interested agents interact to trade information, goods or services; consequently incentive issues arise naturally in such systems.

Moreover, automated decision making, faster computers, low transaction cost and shrinking distances allow the trading parties involved to be strategic and attempt to "game" the system in their favor. In addition, advances in technology have dramatically reduced the cost of storing, retrieving and distributing information. Thereby, reducing the cost of strategic actions.

Fortunately, the technology which allows individual agents to be strategic also gives the principal (i.e. system designer) the ability to implement trading systems with ever more sophisticated rules.

It is, therefor, clear that strategic analysis is a critical step in the design on e-commerce system. *Game Theory* and, in particular, *Mechanism Design Theory* allows us to systematically analyze many of the incentive issues arising in e-commerce systems. In particular, it allows us to select from among many alternatives a system that optimize a given objective while taking into account that self interested agents will all be looking to optimize their own welfare.

To conclude, we quote Greenwald [30]:

> "The interplay of game theory and e-commerce is an exciting domain for future research. Progress in this area will require a combination of theoretical analysis, empirical studies, and simulation experiments. Better market designs will do a better job of matching buyers with sellers, ultimately enhancing the welfare of our society."

In this thesis, we study three problems arising from the strategic interactions of self interested agents in e-commerce settings. In rest of this chapter, we would discuss the context of these three problems, their importance in electronic commerce and our contributions towards a complete solution of these problems. We also give a brief review of Mechanism Design in Appendix A.

1.1 Procurement Auctions with Capacitated Suppliers

In Chapter 2, which is based on Iyengar and Kumar [36], we study the problem of designing expected revenue maximizing procurement auctions with divisible quantities in a setting where both the marginal cost and the production capacity are private information of the suppliers.

Using auctions to award contracts is now pervasive across many industries, see Naegelen [54]; Dasgupta and Spulber [18]; Chen [13] and references therein. The use of procurement auctions to award contracts has been vigorously advocated because competitive bidding results in lower procurement costs, facilitates demand revelation, allows order quantities to be determined ex-post based on the bids and limits the influences of nepotism and political ties. Moreover, the advent of the Internet has significantly reduced the transaction costs involved in conducting such auctions. There is now a large body of literature detailing the growing importance of procurement auctions in industrial procurement. Parente et al. [55]

report that the total value of the B2B online auction transactions totaled 109 billion in 1999, and projected an excellent growth rate of the same.

Since in these auctions, the auctioneer is the *buyer* the bidders are the suppliers or *sellers* and the object being auctioned is the right to supply, these auctions are also called *reverse* auctions.

Although auction design is a well-studied problem (see Dasgupta and Spulber [18]; Ankolekar et al. [4]; Chen [13]; Che [12]), the models analyzed thus far do not adequately address the fact that the private information of the bidders is typically multi-dimensional (cost, capacity, quality, lead times, etc.) and the instruments available to the auctioneer to screen this private information is also multidimensional, e.g. multiple products, multiple components, different procurement locations, etc. In particular, in these models the suppliers are assumed to be uncapacitated; consequently, these auction mechanisms award the contract to a single supplier. In Chapter 2 we investigates mechanism design for a one-shot reverse auction with divisible goods and suppliers with finite capacities in single and multi-product environments. The production capacity, in addition to the production cost, is only known to each of respective suppliers and need to screened by an appropriate mechanism. Thus, in our model the private information of the supplier is two dimensional.

Our main contributions in Chapter 2 are as follows.

1. We introduce a new one-shot procurement auction model where the suppliers have finite privately-known capacity. We show how to construct the optimal direct mechanism for this model with 2-dimensional private information. Although the general direct mechanism design problem with 2-dimensional types is known to be hard, we are able to exploit specific structure of the model to circumvent the difficulties in the general problem. The basic insight is that the optimal mechanism does not give any information

rent[1] to a supplier for revealing capacity information when the production cost is known.

2. We extend our model to multiple product/component procurement with one-dimensional privately known cost parameter and privately known box-shaped capacities. This multi-product extension is similar to Gallien and Wein [29], but their mechanism is neither optimal nor truth revealing. We also present an application of this model to a full commitment multi-period single product model that allows buyers to inter-temporally hedge supply-side capacity risk by ex-post allocating different quantities to different suppliers in different periods.

Results presented in the Chapter 2 are applicable to a number of principle-agent mechanism design problems where the agents have privately known upper bound on the allocations. Examples of such problems include

- Monopoly pricing with adverse selection,
- Scheduling with private values and private deadlines, and
- Knapsack problems with private values and private weights.

1.2 Sponsored Search Auctions

In Chapter 3, which is based on Iyengar and Kumar [35], we design auctions for selling keywords in online sponsored search. Such auctions are used by search service providers such as Google, Yahoo! and MSN for sponsored search advertising, i.e. selling advertisements that appear with their search results. Fain and

[1] The surplus allocated to bidders to provide incentives to reveal their private information is called information rent.

Pedersen [22] and Edelman et al. [21] provides excellent perspectives on how the market mechanisms for sponsored search have evolved over time.

Sponsored search advertising is a major source of revenue for internet search engines. Close to 98% of Google's total revenue of $7.14 billion for the year 2006 came from sponsored search advertisements. It is believed that more than 50% of Yahoo!'s revenue of $6.4 billion in the year 2006 was from sponsored search advertisement. Sponsored search advertisements work as follows. A user queries a certain *adword*, i.e. a keyword relevant for advertisement, on an online search engine. The search engine returns the links to the most "relevant" webpages and, in addition, displays certain number of relevant sponsored links in certain fixed "slots" on the result page. For example, when we search for "`Delhi`" on Google, in addition to the most relevant webpages, eight sponsored links are also displayed which include e.g. links to websites of the hotels in Delhi. Every time the user clicks on any of these sponsored links, she is taken to the website of the advertiser sponsoring the link and the search engine receives certain price per click from the advertiser. It is reasonable to expect that, all things being equal, a user is more likely click on the link that is placed in a slot that is easily visible on the page. In any case, the click likelihood is a function of the slot, and therefore, advertisers have a preference over which slot carries their link and are willing to pay a higher price per click when placed on a more desirable slot. The chance that a user clicks on a sponsored link is likely to be an increasing function of the exogenous brand values of the advertisers; therefore, search engines prefer allocating more desirable slots to advertisers with higher exogenous brand value. In conclusion, the search engines need a mechanism for allocating slots to advertisers. Since auctions are very effective mechanisms for revenue generation and efficient allocation, they have become the mechanism of choice for assigning sponsored links to advertising slots.

The business model used in practice for sponsored search auctions is based on *price-per-click* pricing scheme, where the price charged (which is determined by the auction) to the advertiser by the search engine is proportional to the number of user clicks on his or her advertizement. Alternative price schemes are *price-per-impression* where the payment is proportional to the number of times the advertizement appear on the user screen and *price-per-conversion* where the number of actual sales after the user visit the advertiser website and actually complete a transactions.

We focus on the static models for selling single adwords using dominant strategy equilibrium concept.

Our contributions are as follows:

(a) We formulate a general model for "pay-per-click" adword auctions where the privately known valuation-per-click of advertisers is a function of the allocated slot as well as the identity of the advertiser. Thus, the advertisers have a multi-dimensional type-space. We characterize the set of all dominant strategy incentive compatible, individually rational allocation rules. Using this characterization, we show that there exist incentive compatible allocation rules that are *not* affine maximizers (see Example 3.1).

(b) We consider the special case where the valuation-per-click is slot independent. For this model, we completely solve the mechanism design problem i.e. we characterize of the set of all dominant strategy incentive compatible, individually rational allocation rules, the unique prices that implement these rules, the revenue maximizing mechanism and computationally tractable implementation.

We also analyze rank-based allocations[2] rules and show how to compute an *optimal* rank based allocation rule. We show that even when the click-through-rate matrix is separable[3] the efficiency maximizing rank vector and revenue maximizing rank vector are not same, moreover, the unconstrained revenue maximizing mechanism is *not* rank based (see Example 3.2).

We also propose a new, easy to implement allocation rule that we call the customized rank based allocation rule. We show that this new rule has significantly superior performance – both in terms of efficiency and revenue – when compared with rank based mechanisms.

(c) We analyze a model in which the valuations are privately known constant up to a privately known slot and zero thereafter, also called the slotted valuation model. Such a model is of interest to advertisers who value impressions in addition to the clicks generated, and therefore, desire their advertisement to be placed in highly visible slots only. We present two suboptimal (revenue maximizing) mechanisms for this model, which perform remarkably well on a set of synthetic data. See § 3.4 for details.

We implement all the proposed mechanisms and test their relative performance on a set of synthetic data.

1.3 Equilibrium in Dynamic Online Exchange

In Chapter 4, which is based on Iyengar and Kumar [37], we study a simple dynamic model for matching impatient demand with patient supply over time. By patient supply we mean that sellers are strategic about the amount of time they

[2]In rank based allocation the slots are allocated according to the rankings of weighted bids where highest weighted bid is allocated the top slot and so on.

[3]A matrix $\mathbf{c}$ is called separable iff c_{ij} is of the form $c_{ij} = \phi_i \mu_j$ for all i and j.

are willing to wait for a potential buyer and by impatient demand we mean that buyers are unwilling to wait for potentially better trades in the future.

Strategic models for dynamic economies have had limited success in predicting the real world market dynamics mainly because of the following two reasons: First, it is difficult to analyze a dynamic market with complex dynamics, and consequently models that are solvable are simple abstractions that are unable to incorporate many important features of "real" markets; second, in order to remain tractable economic models need to assume that agents are rational, i.e. agents take the optimal action conditional on the entire market state. Rationality, thus, imposes such unreasonable demands that agents are almost never rational in "real" markets.

To overcome these issues, many models (see, e.g. Daniels et al. [17]; Farmer et al. [23]; Farmer and Zovko [24]; Luckock [47]) for dynamic markets assume that the actions of the agents are randomly distributed according to a distribution that is chosen to reflect nominal economic behavior of agents. In such models, the interesting features of the statistical behavior of the market is a consequence of the market dynamics itself.

In Chapter 4, we present a model that incorporates features of both the strategic and the random market model. In our model, the agents are strategic, i.e. they do not take random actions; however, agents have bounded rationality, and therefore, do not base their actions on the detailed market conditions at the arrival epoch but on the average long term market characteristics arising from the random actions of al agents. In effect, we define an equilibrium concept on the *distribution* induced by the random agent actions. In this equilibrium, in the long run the essential market information filters to the agents and gets reflected in the statistical properties of the market.

Our contribution in Chapter 4 are summarized below.

(a) We define an equilibrium concept on the supply in a dynamic exchange where agents react to the long-term average impact of the actions of all the agents in the exchange. We show that such an equilibrium always exists in our model and show how to numerically compute it. We also provide closed form expressions for various statistical properties of the market, such as the expected time to execution, the average outstanding inventory and the average outstanding inventory conditional on current lowest selling price. Our model is a good approximation to online used book market at *amazon.com*, where multiple copies of substitutable products are available at different prices and sellers strategically post their selling prices based on their beliefs about the execution time.

(b) We show that our proposed equilibrium can be computed in closed form in a market where buyers are not price sensitive. In this solution, the trade execution prices exhibit power tails distribution that matches the empirically observed distribution limit-book driven financial markets.

(c) We extend the our model (as described in (a) above) to multiple competing exchanges indexed by quality of the product traded. In this setting we numerically demonstrate that the equilibrium shows market segmentation along price, quality and cost, where some exchanges serves to low price, high frequency buyers and other exchanges serves to high price but rare buyers.

We believe that the equilibrium notion proposed in this chapter can potentially be applied in other dynamic market clearing mechanisms such as, order driven financial market where both sides of the market are patient and both buyers and sellers queue up and service markets where buyer (sellers) are segmented based on attributes other than price and sellers (buyers) strategically assign resources to each segment.

Chapter 2

Optimal Procurement Auctions with Capacitated Suppliers

2.1 Background and Motivation

Awarding contracts via auctioning is now pervasive across many industries, e.g. electronics industry procurements, government defence procurements, and supply chain procurements. Since the auctioneer is the *buyer*, the bidders are the *suppliers* or *sellers*, and the object being auctioned is the right to supply, these auctions are also called *reverse auctions*. The use of reverse auctions to award contracts has been vigorously advocated since competitive bidding results in lower procurement costs, facilitates demand revelation, allows order quantities to be determined ex-post based on the bids and limits the influences of nepotism and political ties. Moreover, the advent of the Internet has significantly reduced the transaction costs involved in conducting such auctions. There is now a large body of literature detailing the growing importance of reverse auctions in industrial procurement. Parente et al. [55] report that the total value of the B2B online auction transactions totaled 109 billion in 1999, and projected an excellent growth rate of the same.

Although auction design is a well-studied problem, the models analyzed thus far do not adequately address the fact that the private information of the bidders is typically multi-dimensional (cost, capacity, quality, lead times, etc.) and the instruments available to the buyer, i.e. the mechanism designer, to screen this private information is also multidimensional (multiple products, multiple components, different procurement locations, etc). This chapter investigates mechanism design for a one-shot reverse auction with divisible goods and suppliers with finite capacity in single and multi-product environments. The production capacity, in addition to the production cost, are only known to the respective suppliers and need to screened by an appropriate mechanism. Thus in our model, the private information of the supplier is two dimensional.

The model in this chapter is similar to Chen [13] and Dasgupta and Spulber [18], except for the fact that in our model the suppliers have finite production capacity. We refer to suppliers with finite capacity as *capacitated suppliers*.

This chapter is organized as follows. We discuss the relevant literature in § 2.1.1. In § 2.2 we describe the model preliminaries. In § 2.3 we present the analysis for single product optimal direct auction mechanism and it's implementation via "pay as you bid" reverse auction. In § 2.4 we present a simple extension to the multi-product/component model, where the private information about the production cost of the supplier is modeled as one dimensional scalar quantity. In § 2.5 we discuss some of the limitation of the model and directions for future research.

2.1.1 Literature Review

The literature relevant to this problem can be primarily categorized into the following two categories.

2.1.1.1 Operations Management

There is some previous work on supply chain models with finite capacities. Benson [8] investigates optimal allocation in a multi-component environment where the buyer is simply a price-taker, i.e. the buyer pays whatever the suppliers bid. It is easy to convince oneself that in such an environment the suppliers will distort their bids, resulting in very low profits for the buyer.

Swaminathan et al. [66] study the effect of sharing supplier capacity information on the channel profit and profits to individual entities in a model with one manufacturer and two suppliers that differ in cost and capacity. They conclude that information sharing is beneficial to overall supply chain performance; however, it can be detrimental to individual suppliers. It follows, therefor, that unless the suppliers are given proper incentives they are unlikely to reveal their privately known capacity and production cost, and the predicted improvement in channel profits will not be realized. The model investigated in Gallien and Wein [29] is similar to the multi-product model in this chapter. They propose a multi-round mechanism that is neither incentive compatible nor is it optimal. It is difficult to justify that in the proposed multi-round mechanism, where the outcome is determined only in the final round, the suppliers do not have the incentive to deviate from the *myopic best response* (MBR) strategy. Also the results in Gallien and Wein [29] rely on linear programming duality and complimentary slackness; therefore, they may not generalize to more general cost structures.

There is a growing body of work on applying mechanism design techniques to study decentralized decision making and contract design in supply chain management. Deshpande and Schwarz [19] consider an asymmetric information model with single supplier and many retailers and the retailer's order are influenced by privately known demand. They design pricing and allocation (in case of shortage) mechanisms that ensure that the retailers reveal the demand information truthfully. Zhang [74] considers a reverse auction in which the buyer's profit also depends on the lead time offered by the supplier. This paper constructs a mechanism in which the buyer discriminates between suppliers by both their posted lead time as well as the available inventory. However, only the marginal cost information is private, i.e. the supplier type is one dimensional. Chen et al. [14] design a *Vickrey-Clark-Groves* (VCG) mechanism for a supply chain reverse auction with transportation costs. Since the profit to the principal in an efficient auction (i.e. a social welfare maximizing auction) can be arbitrarily smaller than the profit in a revenue maximizing auction, the principal has an incentive to distort information provided to the *third party* auctioneer. For this reason, Chen et al. [14] provide three different auction formats to investigate the relative distortion of the information provided by the bidders and analyze its impact on realized channel profit. The model in this paper assumes that the transportation costs are common knowledge; thus, the agent type space is again one-dimensional. Using a number of simple models, Jin and Wu [39] show that auctions are an effective mechanism for coordinating supply chains. Beil and Wein [7] consider a manufacturer who uses a reverse auction to award a contract to a single supplier based on both prices and a set of non-price attributes that directly affect the valuations. Ankolekar et al. [4] study the design of optimal supply contract when the buyer order is determined after the demand realization, but production as a function of winning supplier cost is determined before demand realization.

2.1.1.2 Microeconomic Theory

In this section we review the microeconomic theory literature on optimal mechanism design with multi-dimensional type or multi-dimensional screening instrument space, which is relevant to this work. Myerson [53] first used the indirect utility approach to characterize the optimal auction in an *independent private value*

(IPV) model. Che [12] considers 2-dimensional (reverse) auction where the sellers bid price and quality and the principal's preference is over both quality and price. However, only the costs are private information and the quality preferences are common knowledge; thus, the bidder type space is one-dimensional. Also, Che [12] only considers sourcing from a single supplier. This leads to a considerable simplification since it reduces the problem to one of determining the winning probability instead of the expected allocation. Naegelen [54] models reverse auctions for department of defense (DoD) projects by a model where the quality of each of the firms are fixed and common knowledge. The preference over quality in this setting results in virtual utilities which are biased across suppliers. Again she only consider single winner case.

Dasgupta and Spulber [18] consider a model very similar to the one discussed in this chapter except that the suppliers have unlimited capacity. They construct the optimal auction mechanism for both single sourcing and multiple sourcing (due to non-linearities in production costs) when the private information is one-dimensional. Chen [13] presents an alternate two-stage implementation for the optimal mechanism in Dasgupta and Spulber [18]. In this alternate implementation the winning firm is first determined via competition on fixed fees, and then the winner is offered an optimal price-quantity schedule.

Laffont et al. [41] solve the optimal nonlinear pricing (single agent principal-agent mechanism design) problem with a two-dimensional type space. They explicitly force the integrability conditions on the gradient of the indirect utility function. Surprisingly, the optimal pricing mechanism (the bundle menus) is rather involved even in the simple setting with a *uniform* prior distribution. Rochet and Stole [60] also provide an excellent survey of multi-dimensional screening and the associated difficulties. In appendix B, we discuss a reverse auction model with capacitated suppliers having convex quadratic costs, where both linear coefficient (representing scale) and the quadratic coefficient (representing capacity) are privately known. We describe the associated optimization problem to explain the complexities that arise with multi-dimensional types.

Vohra and Malakhov [69] describes the indirect utility approach in discrete type-spaces. They re-derive many of the existing results for auctions using network flow techniques and consider optimal auctions with multidimensional types. In Vohra and Malakhov [70], the authors use the same techniques with discrete types-space in an multi-unit optimal auction model where the bidders have privately know capacities in addition to the privately known marginal values. In contrast to Vohra and Malakhov [70],

i) we consider *variable quantity reverse auctions* with *continuous type space*, which allows us to work with more general utility structures;

ii) characterize the set of all incentive compatible mechanisms without assuming monotone allocations;

iii) present an *ironing procedure* under which the optimal mechanism can be characterized under milder regularity conditions;

iv) present a low bid implementation of our optimal direct mechanism and

v) give extensions to capacitated multi-product model.

Voicu [71] consider the procurement auction in a dynamic environment, where bidder takes in to account the possible outcomes of future auctions in a dynamic programming framework.

2.2 Procurement Auctions with Finite Supplier Capacities

We consider a single period model with one buyer (retailer, manufacturer, etc.) and n suppliers. The buyer purchases a single commodity from the suppliers and resells it in the consumer market. The buyer receives an expected revenue, $R(q)$ from selling q units of the product in the consumer market – the expectation is over the random demand realization and any other randomness involved in the downstream market for the buyer that is not contractible. Thus, the side-payment to the suppliers cannot be contingent on the demand realization. We assume $R(q)$ is strictly concave with $R(0) = 0$, $R'(0) = \infty$ and $R'(\infty) = 0$, so that quantity ordered by the buyer is non-zero and bounded. Without this assumption the results in this chapter would remain qualitatively the same; however, the optimal mechanism would have a reservation cost above which the buyer will not order anything. Characterizing the optimal reserve cost is straightforward and is well-studied (see Dasgupta and Spulber [18]).

Supplier i, $i = 1, \ldots, n$, has a constant marginal production cost $c_i \in [\underline{c}, \bar{c}] \subset (0, \infty)$ and finite capacity $q_i \in [\underline{q}, \bar{q}] \subset (0, \infty)$. The joint distribution function of marginal cost c_i and production capacity q_i is denoted by F_i. We assume that (c_i, q_i) and (c_j, q_j) are independently distributed when $i \neq j$, i.e. our model is an *independent private value* (IPV) model. We assume that distribution functions $\{F_i\}_{i=1}^n$ are common knowledge; however, the realization (c_i, q_i) is only known to supplier i. The buyer seeks a revenue maximizing procurement mechanism that ensures that all suppliers participate in the auction.

We employ the direct mechanism approach, i.e. the buyer asks suppliers to directly bid their private information (c_i, q_i). The revelation principle (Myerson [see 53]; Harris and Townsend [see 33]) implies that for any given mechanism one can construct a direct mechanism that has the same point-wise allocation and transfer payment as the given mechanism. Since both mechanisms result in the same expected profit for the buyer, it follows that there is no loss of generality in restricting oneself to direct mechanisms.

We denote the true type of supplier by $\mathbf{b}_i = (c_i, q_i)$ and the supplier i's bid by $\widehat{\mathbf{b}}_i = (\hat{c}_i, \hat{q}_i)$. Let $\mathbf{b} = (\mathbf{b}_1, ..., \mathbf{b}_n)$ and $\widehat{\mathbf{b}} = (\widehat{\mathbf{b}}_1, ..., \widehat{\mathbf{b}}_n)$. Let $\mathbf{B} \equiv \left([\underline{c}, \bar{c}] \times [\underline{q}, \bar{q}]\right)^n$ denote the type space. A procurement mechanism consists of

1. an allocation function $\mathbf{x} : \mathbf{B} \to \mathbb{R}^n_+$ that for each bid vector $\widehat{\mathbf{b}}$ specifies the quantity to be ordered from each of the suppliers, and

2. a transfer payment function $\mathbf{t} : \mathbf{B} \to \mathbb{R}^n$ that maps each bid vector $\widehat{\mathbf{b}}$ to the transfer payment from the buyer to the suppliers.

The buyer seek an allocation function $\mathbf{x}$ and a transfer function $\mathbf{t}$ that maximizes the ex-ante expected profit

$$\Pi(\mathbf{x}, \mathbf{t}) \equiv \mathbb{E}_{\mathbf{b}} \left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b}) \right) - \sum_{i=1}^{n} t_i(\mathbf{b}) \right]$$

subject to the following constraints.

1. feasibility: $x_i(\mathbf{b}) \leq q_i$ for all $i = 1, \ldots, n$, and $\mathbf{b} \in \mathbf{B}$,

2. *incentive compatibility* (**IC**): Conditional on their beliefs about the private information of other bidders, truthfully revealing their private information is weakly dominant for all suppliers, i.e.

$$(c_i, q_i) \in \operatorname*{argmax}_{\substack{\hat{c}_i \in [\underline{c}, \bar{c}] \\ \hat{q}_i \in [\underline{q}, q_i]}} \mathbb{E}_{\mathbf{b}_{-i}} \left\{ t_i((\hat{c}_i, \hat{q}_i), \mathbf{b}_{-i}) - c_i x_i((\hat{c}_i, \hat{q}_i), \mathbf{b}_{-i}) \right\}, \qquad i = 1, \ldots, n, \tag{2.1}$$

The above definition of incentive compatibility is called Bayesian incentive compatibility (see Appendix A). Note that the range for the capacity bid $\hat{q}_i$

is $[\underline{q}, q_i]$, i.e. we do not allow the supplier to overbid capacity. This can be justified by assuming that the supplier incurs a heavy penalty for not being able to deliver the allocated quantity.

3. *individual rationality* (**IR**): The expected interim surplus of each supplier firm is non-negative, for all $i = 1, \ldots, n$, and $\mathbf{b} \in \mathbf{B}$, i.e.

$$\pi_i(\mathbf{b}_i) \equiv \mathbb{E}_{\mathbf{b}_{-i}}\left[t_i(\mathbf{b}) - c_i x_i(\mathbf{b})\right] = T_i(c_i, q_i) - c_i X_i(c_i, q_i) \geq 0. \tag{2.2}$$

Here we have assumed that the outside option available to the suppliers is constant and is normalized to zero.

In this chapter, we use **IC** and **IR** as a shorthand for the incentive compatibility and individual rationality, respectively; and we mean Bayesian incentive compatibility and Bayesian individual rationality, unless specified otherwise.
For any procurement mechanism $(\mathbf{x}, \mathbf{t})$, the *offered* expected surplus $\rho_i(\hat{c}_i, \hat{q}_i)$ when supplier i bids $(\hat{c}_i, \hat{q}_i)$ is defined as follows

$$\rho_i(\hat{c}_i, \hat{q}_i) = T_i(\hat{c}_i, \hat{q}_i) - \hat{c}_i X_i(\hat{c}_i, \hat{q}_i)$$

The offered surplus is simply a convenient way of expressing the expected transfer payment. The expected surplus $\pi_i(c_i, q_i)$ of supplier i with true type (c_i, q_i) when she bids $(\hat{c}_i, \hat{q}_i)$ is given by

$$\pi_i(c_i, q_i) = T_i(\hat{c}_i, \hat{q}_i) - c_i X_i(\hat{c}_i, \hat{q}_i) = \rho_i(\hat{c}_i, \hat{q}_i) + (\hat{c}_i - c_i) X_i(\hat{c}_i, \hat{q}_i).$$

The true surplus π_i equals the offered surplus ρ_i if the mechanism $(\mathbf{x}, \mathbf{t})$ is **IC**.

To further motivate the procurement mechanism design problem, we elaborate on a supplier's incentives to lie about capacity and then consider some illustrative special cases.

2.2.1 Incentive to Underbid Capacity

In this section we show that auctions that ignore the capacity information are not incentive compatible. In particular, the suppliers have an incentive to underbid capacity.

Suppose we ignore the private capacity information and implement the classic K^{th} price auction where the marginal payment to the supplier is equal to the marginal cost of the first losing supplier, i.e. lowest cost supplier among those that did not receive any allocation. Then truthfully bidding the marginal cost is a dominant strategy. However, we show below that in this mechanism the suppliers have an incentive to underbid capacity. Underbidding creates a fake shortage resulting in an increase in the transfer payment that can often more than compensates the loss due to a possible decrease in the allocation. The following example illustrates these incentives in dominant strategy and Bayesian framework.

Example 2.1. *Consider a procurement auction with three capacitated suppliers implemented as the K^{th} price auction. Let $\underline{c} = 1, \bar{c} = 5, \underline{q} = .01$ and $\bar{q} = 6$. Suppose the capacity realization is $(q_1, q_2, q_3) = (5, 1, 5)$ and the marginal cost realization is $(c_1, c_2, c_3) = (1, 1, 5)$. Suppose the buyer wants to procure 5 units and that the spot price, i.e. the outside publicly known cost at which the buyer can procure unlimited quantity is equal to 10. (We need to have an outside market when modeling fixed quantity auction because the realized total capacity of the suppliers can be less than the fixed quantity that needs to be procured.)*

Assume that suppliers 2 and 3 bid truthfully. Consider supplier 1. If she truthfully reveals her capacity, her surplus is \$0; however, if she bids $\hat{q}_1 = 4 - \epsilon$, her surplus is equal to \$$9(4 - \epsilon)$. Thus, bidding truthfully is not a dominant strategy for supplier 1.

Next, we show that for appropriately chosen asymmetric prior distributions supplier 1 has incentives to underbid capacity even in the Bayesian framework. Assume that the marginal cost and capacity are independently distributed. Let $(c_1, q_1) = (1, 5)$. Thus

$\mathbb{P}((1 \leq c_2) \cap (1 \leq c_3)) = 1$. *Let the capacity distribution* F_i^q, $i = 2, 3$, *be such that* $\mathbb{P}(q_2 + q_3 \leq 1) > 1 - \epsilon$ *for some* $0 < \epsilon \ll 1$. *Then the expected surplus* $\pi_1(1, 5)$, *if supplier* 1 *bids her capacity truthfully, is upper bounded by* $5 \times (\bar{c} - 1) = 20$. *On the other hand the expected surplus if she bids* $4 - \epsilon$ *is lower bounded by* $9 \times (4 - \epsilon) \times (1 - \epsilon)$. *Thus, supplier* 1 *has ex-ante incentive to underbid capacity.*

Figure 2.1 shows two uniform price auction mechanisms, the K^{th} price auction and the market clearing mechanism. In our model, the suppliers can change the supply ladder curve both in terms of location of the jumps (by misreporting costs) and the magnitude of the jump (by misreporting capacity). We know that in a model with commonly known capacities, the fixed quantity optimal auction can be implemented as K^{th} price auction. We showed in the example above that in the K^{th} price auction with privately known capacity, the suppliers can "*game*" the mechanism.

This effect is also true if prices are determined by the market clearing condition. Suppose the suppliers truthfully reveal their marginal costs and the buyer aggregates these bids to form the supply curve $Q(p) = \sum_{i=1}^{n} \hat{q}_i \mathbf{1}_{\{c_i \leq p\}}$. The demand curve $D(p)$ in this context is given by

$$D(p) = \operatorname*{argmax}_{u \geq 0} \left[R(u) - pu \right] = (R')^{-1}(p).$$

Thus, the equilibrium price p^* is given by the solution of the market clearing condition $(R')^{-1}(p) = Q(p^*)$ (see Figure 2.1). The model primitives ensure that the market clearing price $p^* \in (0, \infty)$. In such a setting, as in the K-th price auction, the supplier with low cost and high capacity can at times increase surplus by underbidding capacity because the increase in the marginal (market clearing) price can offset the decrease in allocation.

The above discussion shows that both the K-th price auction and the market-clearing mechanism are not truth revealing. In § 2.2.2.3 we show that if the suppliers bid the cost truthfully for exogenous reasons, the buyer can extract all the

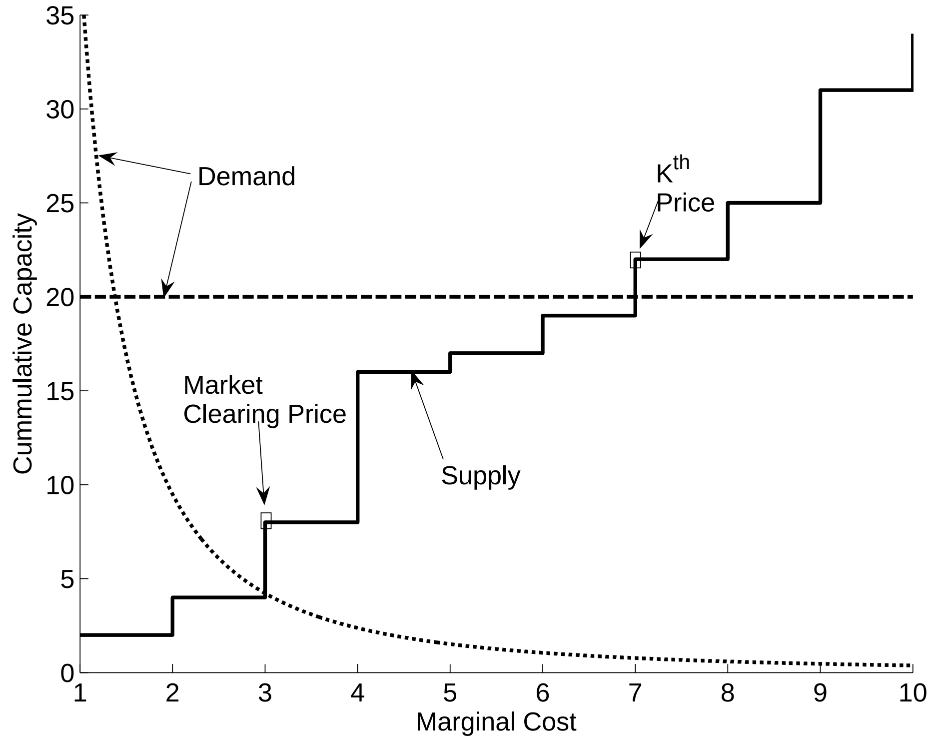


Figure 2.1: Uniform price auctions: K^{th} price auction and market clearing price auction

surplus, i.e. the buyer does not pay any information rent to the suppliers for the capacity information. In this mechanism the transfer payments are simply the true costs of the supplier and the quantity allocated is a monotonically decreasing function of the marginal cost. This optimal mechanism is *discriminatory* and unique. In particular, with privately known capacities, there does not exist a *uniform price* optimal auction. Ausubel [5] shows that a modified market clearing mechanism, where items are awarded at the price that they are "clinched", is efficient, i.e. socially optimal (Ausubel and Cramton [see, also 6]).

2.2.2 Relaxations

In this section we discuss some special cases of the procurement mechanism design problem formulated in § 2.2.

2.2.2.1 Full Information (or First-best) Solution

Suppose all suppliers bid truthfully. It is clear that in this setting the surplus of each supplier would be identically zero. Denote the marginal cost of supplier firm with i^{th} lowest marginal cost by $c_{[i]}$ and it's capacity by $q_{[i]}$. Then the piece-wise convex linear cost function faced by the buyer is given by

$$c(y) = \sum_{j=1}^{i-1} q_{[j]} c_{[i]} + \left(y - \sum_{j=1}^{i-1} q_{[j]} \right) c_{[i]} \quad \text{for } \sum_{j=1}^{i-1} q_{[j]} \leq y \leq \sum_{j=1}^{i} q_{[j]} \tag{2.3}$$

The optimal procurement strategy for the buyer is the same as that of a buyer facing a single supplier with piece wise linear convex production cost $c(y)$. Clearly, multi-sourcing is optimal with a number of lowest cost suppliers producing at capacity and at most one supplier producing below capacity.

Multiple sourcing can also occur in an uncapacitated model when the production costs are nonlinear. We expect that a risk averse buyer would also find it advantageous to multi-source to diversify the ex-ante risk due to the asymmetric information. Since, to the best of our knowledge, the problem of optimal auctions with a risk averse principal has not been fully explored in the literature, this remains a conjecture.

2.2.2.2 Second-degree Price Discrimination with a Single Capacitated Supplier

Suppose there is a single supplier with privately known marginal cost and capacity. Suppose the capacity and cost are independently distributed. Let $F(c)$ and $f(c)$ denote, respectively, the cumulative distribution function (CDF) and density of the marginal cost c and suppose the hazard rate $\frac{f(c)}{F(c)}$ is monotonically decreasing, i.e. we are in the so-called regular case (Myerson [53]). Note that this is a standard adverse selection problem (see chapter 2 and 3 in Salanie [63]); the procurement counterpart of second degree price discrimination in the monopoly pricing model.

We will first review the optimal mechanism when the supplier is uncapacitated. Using the indirect utility approach, the buyer's problem can formulated as follows.

$$\max_{\substack{x(\cdot)\geq 0\\ x(\cdot)\text{monotone}}} \mathbb{E}_c\left[R(x(c)) - \left(c + \frac{F(c)}{f(c)}\right)x(c)\right]. \tag{2.4}$$

Let $x^*(c)$ denote the optimal solution of the relaxation of (2.4) where one ignores the monotonicity assumption, i.e.

$$x^*(c) \in \operatorname*{argmax}_{x\geq 0}\left\{R(x) - \left(c + \frac{F(c)}{f(c)}\right)x\right\}.$$

Then, regularity implies that x^* is a monotone function of c, and is, therefore, feasible for (2.4). The transfer payment $t^*(c)$ that makes the optimal allocation x^* **IC** is given by

$$t^*(c) = cx^*(c) + \int_c^{\bar{c}} x^*(u)du.$$

Since the optimal allocation $x^*(c)$ and the transfer payment $t^*(c)$ are both monotone in c, the cost parameter c can be eliminated to obtain the transfer t directly in terms of the allocation x, i.e. a *tariff* $t^*(x)$. The indirect tariff implementation is very appealing for implementation as it can "posted" and the supplers can simply self-select the production quantity based on the posted tariff.

Now consider the case of a capacitated supplier. Feasibility requires that for all $c \in [\underline{c}, \bar{c}]$, $0 \leq x(c) \leq q$. Suppose the supplier bids the capacity truthfully. (We justify this assumption below.) Then the buyer's problem is given by

$$\max_{\substack{x(\cdot,\cdot)\geq 0\\ x(\cdot,q)\text{monotone}}} \mathbb{E}_{(c,q)}\left[R(x(c,q)) - \left(c + \frac{F(c)}{f(c)}\right)x(c,q)\right] \tag{2.5}$$

where F denotes the marginal distribution of the cost. Set the allocation $\hat{x}(c,q) = \min\{x^*(c), q\}$, where x^* denotes the optimal solution of the uncapacitated problem (2.4). Then $\hat{x}$ is clearly feasible for (2.5). Moreover,

$$\hat{x}(c,q) \in \operatorname*{argmax}_{0\leq x\leq q}\left\{R(x) - \left(c + \frac{F(c)}{f(c)}\right)x\right\}.$$

Thus, $\hat{x}$ is an optimal solution of (2.5). As before, set transfer payment $\hat{t}(c,q) = c\hat{x}(c,q) + \int_c^{\bar{c}} \hat{x}(u,q)du$. Then, the supplier surplus in the solution $(\hat{x},\hat{t})$ is non-decreasing in the capacity bid q. Therefore, it is weakly dominant for the supplier to bid the capacity truthfully, and our initial assumption is justified. Note that the supplier surplus $\hat{\pi}(c,q) = \int_c^{\bar{c}} \hat{x}(u,q)du$.

The fact that the capacitated solution $\hat{x}(c,q) = \min\{x^*(c),q\}$ is simply a truncation of the uncapacitated solution $x^*(c)$ allows one to implement it in a very simple manner. Suppose the buyer offers the seller the tariff $t^*(x)$ corresponding to the uncapacitated solution. Then the solution $\tilde{x}$ of the seller's optimization problem $\max_{0\leq x\leq q}\{t^*(x) - cx\}$ is given by

$$\tilde{x} = \min\{x^*(c),q\} = \hat{x}(c,q),$$

i.e. the quantity supplied is the same as that dictated by the optimal capacitated mechanism.

Define $c_q = \sup\{c \in [\underline{c},\bar{c}] : x^*(c) \geq q\}$. Then the monotonicity of $x^*(c)$ implies that

$$\tilde{x} = \hat{x}(c,q) = \begin{cases} x^*(c), & c > c_q, \\ q, & c \leq c_q. \end{cases}$$

Then, for all $c > c_q$, the supplier requests $x^*(c)$ and receives a surplus

$$\begin{aligned} \tilde{\pi}(c) = t^*(x^*(c)) - cx^*(c) &= \int_c^{\bar{c}} x^*(u)du \\ = \int_c^{\bar{c}} \min\{x^*(u),q\}du &= \int_c^{\bar{c}} \hat{x}(u,q)du = \hat{\pi}(c,q). \end{aligned}$$

For $c \leq c_q$, the supplier request q and the surplus

$$\begin{aligned} \tilde{\pi}(c) &= t^*(q) - cq, \\ &= t^*(x^*(c_q)) - c_q q + (c_q - c)q, \\ &= \pi^*(c_q) + (c_q - c)q = \int_{c_q}^{\bar{c}} x^*(u)du + \int_c^{c_q} q du = \int_c^{\bar{c}} \hat{x}(u,q)du = \hat{\pi}(c,q). \end{aligned}$$

Thus, the supplier surplus in the tariff implementation is $\hat{\pi}(c, q)$, the surplus associated with optimal capacitated mechanism. Consequently, it follows that the "full" tariff implements the capacitated optimal mechanism! This immediately implies that the buyer does need to know the capacity of the supplier, and pays zero information rent for the capacity information. In the next section we show that the assumption of independence of capacity and cost is critical for this result.

2.2.2.3 Marginal Cost Common Knowledge

Suppose the marginal costs are common knowledge and only the production capacities are privately known. Then the optimal procurement mechanism maximizes

$$\max_{(\mathbf{x},\mathbf{t})} \mathbb{E}_q \left[R\Big(q_i \sum_{i=1}^{n} x_i(q)\Big) - \sum_{i=1}^{n} t_i(q) \right]$$

subject to the constraint that the expected supplier i's surplus $T_i(q_i) - c_i X_i(q_i)$ is weakly increasing in q_i (**IC**) and nonnegative (**IR**) for all suppliers i.

Not surprisingly, the first-best or the full-information solution works in this case. Set the transfer payment equal to the production costs of the supplier, i.e. $t_i(q) = c_i x_i(q)$. Then the supplier surplus is zero and the buyer's optimization problem reduces to the full-information case. Since the full-information allocation $x_i(\hat{q}_i, q_{-i})$ is weakly increasing in $\hat{q}_i$ for all q_{-i}, bidding the true capacity is a weakly dominant strategy for the suppliers. Thus, the buyer can effectively ignore the **IC** constraints above and follow the full information allocation scheme and extract all the supplier surplus. The fact that, conditional on knowing the cost, the buyer does not offer any informational rent for the capacity information is crucial to the result in the next section.

2.3 Characterizing Optimal Direct Mechanism

We use the standard indirect utility approach to characterize all incentive compatible and individually rational direct mechanisms and the minimal transfer payment function that implements a given incentive compatible allocation rule (Lemma 2.1). The characterization of the transfer payment allows us to write the expected profit of the buyer for a given incentive compatible allocation rule as a function of the allocation rule and the offered surplus $\rho_i(\bar{c}, q)$ (Theorem 2.1). To proceed further, we make the following assumption.

Assumption 2.1. *For all $i = 1, 2, \cdots, n$, the joint density $f_i(c_i, q_i)$ has full support.*

Note that if Assumption 2.1 holds then the conditional density $f_i(c_i|q_i)$ also has full support.

Lemma 2.1. *Procurement mechanisms with capacitated suppliers satisfy the following.*

1. *A feasible allocation rule $\mathbf{x} : \mathbf{B} \rightarrow \mathbb{R}^n_+$ is* **IC** *if, and only if, the expected allocation $X_i(c_i, q_i)$ is non-increasing in the cost parameter c_i for all suppliers $i = 1, \ldots, n$.*

2. *A mechanism $(\mathbf{x}, \mathbf{t})$ is* **IC** *and* **IR** *if, and only if, the allocation rule $\mathbf{x}$ satisfies (a) and the offered surplus $\rho_i(\hat{c}_i, \hat{q}_i)$ when supplier i bids $(\hat{c}_i, \hat{q}_i)$ is of the form*

$$\rho_i(\hat{c}_i, \hat{q}_i) = \rho_i(\bar{c}, \hat{q}_i) + \int_{\hat{c}_i}^{\bar{c}} X_i(u, \hat{q}_i) du \tag{2.6}$$

   with $\rho_i(\hat{c}_i, \hat{q}_i)$ non-negative and non-decreasing in $\hat{q}_i$ for all $\hat{c}_i \in [\underline{c}, \bar{c}]$ and i.

Remark 2.1. *Recall that the offered surplus ρ_i is, in fact, equal to the surplus π_i when the allocation rule* $\mathbf{x}$ *(and the associated transfer payment* $\mathbf{t}$*) is* **IC**.

Proof: Fix the mechanism $(\mathbf{x}, \mathbf{t})$. Then the supplier i expected surplus $\pi_i(c_i, q_i)$ is given by

$$\pi_i(c_i, q_i) = \max_{\substack{\hat{c}_i \in [\underline{c}, \bar{c}] \\ \hat{q}_i \in [\underline{q}, q_i]}} \left\{ T_i(\hat{c}_i, \hat{q}_i) - c_i X_i(\hat{c}_i, \hat{q}_i) \right\}. \tag{2.7}$$

Note that the capacity bid $\hat{q}_i \le q_i$, the true capacity. This plays an important role in the proof. From (2.7), it follows that for all fixed $q \in [\underline{q}, \bar{q}]$, the surplus $\pi_i(c_i, q_i)$ is convex in the cost parameter c_i. (There is, however, no guarantee that $\pi_i(c_i, q_i)$ is jointly convex in (c_i, q_i).) Consequently, for all fixed $q \in [\underline{q}, \bar{q}]$, the function $\pi_i(c_i, q_i)$ is absolutely continuous in c and differentiable almost everywhere in c.

Since **x** is **IC**, it follows that (c_i, q_i) achieves the maximum in (2.7). Thus, in particular,

$$c_i \in \operatorname*{argmax}_{\hat{c}_i \in [\underline{c}, \bar{c}]} \left\{ T_i(\hat{c}_i, q_i) - c_i X_i(\hat{c}_i, q_i) \right\}, \tag{2.8}$$

i.e. if supplier i bids capacity q truthfully, it is still optimal for her to bid the cost truthfully. Since $\pi_i(c_i, q_i)$ is convex in c_i, (2.8) implies that

$$\frac{\partial \pi_i(c, q)}{\partial c} = -X_i(c, q), \quad \text{a.e.} \tag{2.9}$$

Consequently, $X_i(c, q)$ is non-increasing in c for all $q \in [\underline{q}, \bar{q}]$. This proves the forward direction of the assertion in part (a).

To prove the converse of part (a), suppose $X_i(c_i, q_i)$ is non-increasing in c_i for all q_i. Set the offered surplus

$$\rho_i(\hat{c}_i, \hat{q}_i) = \bar{\rho}_i(\hat{q}_i) + \int_c^{\bar{c}} X_i(u, \hat{q}_i) du$$

where the function $\bar{\rho}_i(\hat{q}_i) \triangleq \rho(\bar{c}, \hat{q}_i)$ is such that $\rho_i(\hat{c}_i, \hat{q}_i)$ is non-decreasing in $\hat{q}_i$ for all $\hat{c}_i \in [\underline{c}, \bar{c}]$. There are many feasible choices for $\bar{\rho}(\hat{q}_i)$. In particular, if $\frac{\partial X_i(c,q)}{\partial q}$ exists a.e., one can set,

$$\bar{\rho}_i(\hat{q}_i) = \sup_{c_i \in [\underline{c}, \bar{c}]} \left\{ \int_{\underline{q}}^{q_i} \int_{c_i}^{\bar{c}} \left(\frac{\partial X_i(t, z)}{\partial z} \right)^- dt dz \right\}.$$

For any such choice of $\bar{\rho}_i$, the supplier i surplus

$$\begin{aligned}
\pi_i(\hat{c}_i, \hat{q}_i) &= \rho_i(\hat{c}_i, \hat{q}_i) + (\hat{c}_i - c_i)X_i(\hat{c}_i, \hat{q}_i), \\
&= \bar{\rho}_i(\hat{q}_i) + \int_{\hat{c}_i}^{\bar{c}} X_i(u, \hat{q}_i)du + (\hat{c}_i - c_i)X_i(\hat{c}_i, \hat{q}_i), \\
&= \bar{\rho}_i(\hat{q}_i) + \int_{c_i}^{\bar{c}_i} X_i(u, \hat{q}_i)du + \int_{\hat{c}_i}^{c_i} X_i(u, \hat{q}_i)du + (\hat{c}_i - c_i)X_i(\hat{c}_i, \hat{q}_i), \\
&\leq \bar{\rho}_i(\hat{q}_i) + \int_{c_i}^{\bar{c}} X_i(u, \hat{q}_i)du, \qquad (2.10) \\
&\leq \bar{\rho}_i(q_i) + \int_{c_i}^{\bar{c}} X_i(u, q_i)du, \qquad (2.11) \\
&= T_i(c_i, q_i) - c_i X_i(c_i, q_i) = \pi_i(c_i, q_i),
\end{aligned}$$

where (2.10) follows from the fact that $X_i(c, q)$ in non-increasing in c for all fixed q and (2.11) follows from the $\rho_i(\hat{c}_i, \hat{q}_i)$ is non-decreasing in $\hat{q}_i$ and $\hat{q}_i \leq q_i$. Thus, we have established that it is weakly dominant for supplier i to bid truthfully, or equivalently **x** is an incentive compatible allocation.

From (2.9) we have that whenever **x** is **IC** we must have that the supplier surplus is of the form

$$\pi_i(c_i, q_i) = \pi_i(\bar{c}, q_i) + \int_{c}^{\bar{c}} X_i(u, q_i)du.$$

Since **x** is **IR**, $\pi_i(\bar{c}_i, q_i) \geq 0$, and, since **x** is **IC**,

$$q_i \in \underset{\hat{q}_i \leq q_i}{\operatorname{argmax}} \{T_i(c_i, \hat{q}_i) - c_i X_i(c_i, \hat{q}_i)\} = \underset{\hat{q}_i \leq q_i}{\operatorname{argmax}} \{\pi_i(c_i, \hat{q}_i)\}.$$

Thus, we must have that $\pi_i(c_i, q_i)$ is non-decreasing in q_i for all $c_i \in [\underline{c}, \bar{c}]$. This establishes the forward direction of part (b).

Suppose the offered surplus if of the form (2.6) then $(\mathbf{x}, \mathbf{t})$ satisfies **IR**. Since $X_i(c_i, q_i)$ is non-increasing in c_i for all q_i, it follows that $\pi_i(c_i, q_i)$ is convex in c_i for all q_i and $\frac{\partial \pi_i(c_i, q_i)}{\partial c_i} = -X_i(c_i, q_i)$. Consequently,

$$\pi_i(\hat{c}_i, \hat{q}_i) = \rho_i(\hat{c}_i, \hat{q}_i) + (c_i - \hat{c}_i)\big(-X_i(\hat{c}_i, \hat{q}_i)\big) \leq \pi_i(c_i, \hat{q}_i) \leq \pi_i(c_i, q_i),$$

where the last inequality follows from the fact that $\pi_i(c_i, q_i)$ is non-decreasing in q_i for all c_i and $\hat{q}_i \leq q_i$. Thus, we have establishes that $(\mathbf{x}, \mathbf{t})$ is **IC**. ■

Next, we use the results in Lemma 2.1 to characterize the buyer's expected profit.

Theorem 2.1. *Suppose Assumption 2.1 holds. Then the buyer profit $\Pi(\mathbf{x}, \mathbf{t})$ corresponding to any feasible allocation rule $\mathbf{x} : \mathbf{B} \to \mathbb{R}^n_+$ that satisfies* **IC** *and* **IR** *is given by*

$$\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}}) = \mathbb{E}_b \left[R\left(\sum_{i=1}^{n} x_i(b) \right) - \sum_{i=1}^{n} x_i(b) H_i(c_i, q_i) - \sum_{i=1}^{n} \bar{\rho}_i(q_i) \right], \tag{2.12}$$

where $\bar{\rho}_i(q_i)$ is the surplus offered when the supplier i bid is $(\bar{c}, q_i)$ and $H_i(c_i, q_i)$ denotes the virtual costdefined to be $H_i(c_i, q_i) \equiv c_i + \frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}$.

Remark 2.2. *Theorem 2.1 implies that the buyer's profit is determined by* both *the allocation rule* $\mathbf{x}$ *and offered surplus $\bar{\boldsymbol{\rho}}(q)$ when supplier i bid is $(\bar{c}, q)$. We emphasize this by denoting the buyer profit by $\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}})$.*

Proof: From Lemma 2.1, we have that the offered supplier *i* surplus $\rho_i(c_i, q_i)$ under any **IC** and **IR** allocation rule $\mathbf{x}$ is of the form

$$\rho_i(c_i, q_i) = \rho_i(\bar{c}, q_i) + \int_{c_i}^{\bar{c}} X_i(t, q_i) dt$$

Thus, the buyer profit function is

$$\begin{aligned} \Pi = \mathbb{E}_{\mathbf{b}} & \left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b}) \right) - \sum_{i=1}^{n} \left(c_i x_i(\mathbf{b}) + \rho_i(\bar{c}, q_i) \right) \right] \\ & - \sum_{i=1}^{n} \left(\int_{\underline{q}}^{\bar{q}} \int_{\underline{c}}^{\bar{c}} \int_{c_i}^{\bar{c}} X_i(u_i, q_i) du_i f_i(c_i, q_i) dc_i dq_i \right). \end{aligned}$$

By interchanging the order of integration, we have

$$\begin{aligned} \int_{\underline{c}}^{\bar{c}} dc_i f_i(c_i, q_i) \int_{c_i}^{\bar{c}} du_i X_i(u_i, q_i) &= \int_{\underline{c}}^{\bar{c}} du_i X_i(u_i, q_i) \int_{\underline{c}}^{t} dc f_i(c, q_i) \\ &= \int_{\underline{c}}^{\bar{c}} X_i(c_i, q_i) F_i(c_i \mid q_i) f_i(q_i) dc_i. \end{aligned}$$

Substituting this back into the expression for profit, we get

$$\begin{aligned}\Pi(\mathbf{x}) &= \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b})\right) - \sum_{i=1}^{n}\left(c_i x_i(\mathbf{b}) + \rho_i(\bar{c}, q_i)\right)\right] \\ &\quad - \sum_{i=1}^{n}\left(\int_{\underline{q}}^{\bar{q}}\int_{\underline{c}}^{\bar{c}} X_i(c_i, q_i) F_i(c_i|q_i) f_i(q_i) dc_i dq_i\right), \\ &= \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b})\right) - \sum_{i=1}^{n}\left(c_i x_i(\mathbf{b}) + \rho_i(\bar{c}, q_i)\right)\right] - \sum_{i=1}^{n}\mathbb{E}_{\mathbf{b}}\left[x_i(\mathbf{b})\frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}\right] \\ &= \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b})\right) - \sum_{i=1}^{n}\left(c_i + \frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}\right) x_i(\mathbf{b}) - \sum_{i=1}^{n}\rho_i(\bar{c}, q_i)\right].\end{aligned}$$

This establishes the result. ■

The virtual marginal costs $H_i(c, q)$ in our model are very similar to the virtual marginal costs in the uncapacitated reverse auction model; except that the virtual costs are now defined in terms of the distribution of the marginal cost c_i *conditioned* on the capacity bid q_i. Thus, the capacity bid provides information only if the cost and capacity are correlated. Next, we characterize the optimal allocation rule under the regularity Assumption 2.2 and to a limited extent under general model primitives.

2.3.1 Optimal Mechanism in the Regular Case

In this section, we make the following additional regularity assumption about the monotonicity of the virtual marginal costs.

Assumption 2.2 (Regularity). *For all $i = 1, 2, \cdots, n$, the virtual cost function*

$$H_i(c_i, q_i) \equiv c_i + \frac{F_i\left(c_i|q_i\right)}{f_i(c_i|q_i)}$$

is non-decreasing in c_i and non-increasing in q_i.

Assumption 2.2 is called the *regularity* condition. This regularity condition on virtual cost is commonly assumed in literature on procurement auctions, except that we require monotonicity in both the cost variable as well as the capacity variable. It is satisfied when the conditional density of the marginal cost given capacity is log concave in c_i, and the production cost and capacity are, loosely speaking, "negatively affiliated" in such a way that $\frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}$ is non-increasing in q_i. This is true, for example, when the cost and capacity are independent.

For $\mathbf{b} \in \mathbf{B}$, define

$$\mathbf{x}^*(\mathbf{b}) \equiv \underset{0 \le \mathbf{x} \le \mathbf{q}}{\operatorname{argmax}} \left\{ R\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n x_i H_i(c_i, q_i) \right\}, \tag{2.13}$$

where the inequality $0 \le \mathbf{x} \le \mathbf{q}$ is interpreted component-wise. We call $\mathbf{x}^* : \mathbf{B} \to \mathbb{R}^n_+$ the point-wise optimal allocation rule. Since (2.13) is identical to the full information problem with the cost c_i replaced by the *virtual cost* $H_i(c_i, q_i)$, it follows that (2.13) can be solved by aggregating all the suppliers into one meta-supplier. Denote the virtual cost of supplier with i^{th} lowest virtual cost by $h_{[i]}$ and the corresponding capacity by $q_{[i]}$. Then the buyer faces a piece-wise convex linear cost function $h(q)$ given by

$$h(q) = \sum_{j=1}^{i-1} q_{[j]} h_{[i]} + \left(q - \sum_{j=1}^{i-1} q_{[j]}\right) c_{[i]}, \tag{2.14}$$

for $\sum_{j=1}^{i-1} q_{[j]} \le q \le \sum_{j=1}^{i} q_{[j]}$, $i = 1, \ldots, n$, where $\sum_{j=1}^{0} q_{[j]}$ is set to zero. From the structure of the supply curve it follows that the optimal solution of (2.13) is of the form

$$x^*_{[i]} \equiv \begin{cases} q_{[i]}, & [i] < [i]^*, \\ \le q_{[i]}, & [i] = [i]^*, \\ 0 & \text{otherwise}, \end{cases} \tag{2.15}$$

where $1 \le [i]^* \le n$.

Lemma 2.2. *Suppose Assumption 2.2 holds. Let $\mathbf{x}^* : \mathbf{B} \to \mathbb{R}^n_+$ denote the point-wise optimal defined in (2.13).*

1. *$x_i^*((c_i, q_i), \mathbf{b}_{-i})$ is non-increasing in c_i for all fixed q_i and $\mathbf{b}_{-i}$. Consequently, $X_i(c_i, q_i)$ is non-increasing in c_i for all q_i.*

2. *$x_i^*((c_i, q_i), \mathbf{b}_{-i})$ is non-decreasing in q_i for all fixed c_i and $\mathbf{b}_{-i}$. Therefore, $X_i(c_i, q_i)$ is non-decreasing in q_i for all fixed c_i.*

Proof: From (2.15) it is clear that $\mathbf{x}^*((c_i, q_i), \mathbf{b}_{-i})$ is non-increasing in the virtual cost $H_i(c_i, q_i)$. When Assumption 2.2 holds, the virtual cost $H_i(c_i, q_i)$ is non-decreasing in c_i for fixed q_i; consequently, the allocation x_i^* is non-increasing in the capacity bid q_i for fixed c_i and $\mathbf{b}_{-i}$. Part (a) is established by taking expectations of $\mathbf{b}_{-i}$. A similar argument proves (b). ■

We are now in position to prove the main result of this section.

Theorem 2.2. *Suppose Assumption 2.1 and 2.2 hold. Let $\mathbf{x}^*$ denote the point-wise optimal solution defined in (2.13). For $i = 1, \ldots, n$, set the transfer payment*

$$t_i^*(\widehat{\mathbf{b}}) = \hat{c}_i X_i^*(c_i, q_i) + \int_{\hat{c}_i}^{\bar{c}} X_i^*((u, \hat{q}_i))du. \tag{2.16}$$

Then $(\mathbf{x}^*, \mathbf{t}^*)$ is Bayesian incentive compatible revenue maximizing procurement mechanism.

Proof: From (2.12), it follows that the buyer profit

$$\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}}) \leq \mathbb{E}_{\mathbf{b}}\left[\max_{\mathbf{0}\leq\mathbf{x}\leq\mathbf{q}}\left\{R\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} x_i H_i(c_i, q_i)\right\}\right] = \Pi(\mathbf{x}^*, \mathbf{0}).$$

Thus, all that remains to be shown is that the offered surplus ρ_i^* corresponding to the transfer payment $\mathbf{t}^*$ satisfies $\bar{\rho}_i^*(q_i) = \rho_i^*(\bar{c}_i, q_i) \equiv 0$, and $(\mathbf{x}^*, \mathbf{t}^*)$ is **IC** and **IR**. From (2.16), it follows that the offered surplus

$$\rho_i^*(\hat{c}_i, \hat{q}_i) = \int_{\hat{c}_i}^{\bar{c}} X_i^*(u, \hat{q}_i)du. \tag{2.17}$$

Thus, $\bar{\rho}_i^*(q_i) = \rho_i^*(\bar{c}, q_i) \equiv 0$.

Next, Lemma 2.2 (a) implies that $X_i^*((\hat{c}_i, \hat{q}_i), \mathbf{b}_{-i})$ is non-increasing in c_i for all q_i. From Lemma 2.2 (b), we have that $X_i(u, \hat{q}_i)$ is non-decreasing in $\hat{q}_i$. From (2.17), it follows that $\pi_i(c_i, q_i)$ is non-decreasing in q_i for all c_i. Now, Lemma 2.1 (b) allows us to conclude that $(\mathbf{x}^*, \mathbf{t}^*)$ is **IC**.

Since $(\mathbf{x}^*, \mathbf{t}^*)$ satisfies **IC**, the offered surplus $\rho_i^*(c_i, q_i)$ is, indeed, the supplier surplus. Then (2.17) implies that $(\mathbf{x}^*, \mathbf{t}^*)$ is **IR**. ■

Next, we illustrate the optimal reverse auction on a simple example.

Example 2.2. *Consider a procurement auction with two identical suppliers. Suppose the marginal cost c_i and capacity q_i of each of the suppliers are uniformly distributed over the unit square,*

$$f_i(c_i, q_i) = 1 \quad \forall (c_i, q_i) \in [0,1]^2, i = 1, 2.$$

Therefore, the virtual costs

$$H_i(c_i, q_i) = c_i + \frac{F_i(c_i|q_i)}{f_i(c_i|q_i)} = c_i + c_i = 2c_i \quad \forall c_i \in [0,1], i = 1, 2.$$

It is clear that this example satisfies Assumption 2.1 and Assumption 2.2.

Suppose the buyer revenue function $R(q) = 4\sqrt{q}$. Then, it follows that buyer's optimization problem reduces to the point-wise problem

$$\mathbf{x}^*(\mathbf{c}, \mathbf{q}) = \operatorname*{argmax}_{\mathbf{x} \leq \mathbf{q}} \left\{ 4\sqrt{\sum_{i=1}^{2} x_i} - 2\sum_{i=1}^{2} c_i x_i \right\}.$$

The above constrained problem can be easily solved using the Karush-Kuhn-Tucker (KKT) conditions which are sufficient because of strict concavity of the buyer's profit function. For $i = 1, 2$, the solution is given by,

$$x_i^*(c, q) = \begin{cases} \frac{1}{c_i^2} & c_i \leq c_{-i},\ q_i \geq \frac{1}{c_i^2}, \\ q_i & c_i \leq c_{-i},\ q_i < \frac{1}{c_i^2}, \\ 0 & c_i \geq c_{-i},\ q_{-i} \geq \frac{1}{c_{-i}^2}, \\ \min\left\{ \max\left\{ 0, \frac{1}{c_i^2} - q_{-i} \right\}, q_i \right\} & \text{otherwise.} \end{cases}$$

where $-i$, is the index of the supplier competing with supplier i. The corresponding expected transfer payments are given by equation (2.16).

In order for an allocation rule $\mathbf{x}$ to be Bayesian incentive compatible it is sufficient that the expected allocation $X_i(c_i, q_i)$ be weakly monotone in c_i and q_i. Assumption 2.2 ensures that the point-wise optimal allocation x_i^* is weakly monotone in c_i and q_i. This stronger property of $\mathbf{x}^*$ can be exploited to show that $\mathbf{x}^*$ can be implemented in the dominant strategy solution concept, i.e. there exist a transfer payment function under which truth telling forms an dominant strategy equilibrium.

Theorem 2.3. *Suppose Assumption 2.1 and Assumption 2.2 hold. For $i = 1, \ldots, n$, let the transfer payment be*

$$t_i^{**}(\widehat{\mathbf{b}}) = \hat{c}_i x_i^*(\widehat{\mathbf{b}}) + \int_{\hat{c}_i}^{\bar{c}} x_i^*((u, \hat{q}_i), \widehat{\mathbf{b}}_{-i}) du. \tag{2.18}$$

Then, $(\mathbf{x}^*, \mathbf{t}^{**})$ is an dominant strategy incentive compatible, individually rational and revenue maximizing procurement mechanism.

Proof: It is clear that the buyer profit under any dominant strategy **IC** and **IR** mechanism is upper bounded by the profit $\Pi(\mathbf{x}^*, \mathbf{0})$ of the point-wise optimal allocation $\mathbf{x}^*$. From (2.18), it follows that $(\mathbf{x}^*, \mathbf{t}^{**})$ is ex-post (pointwise) **IR**.

Thus, all that remains is to show that $(\mathbf{x}^*, \mathbf{t}^{**})$ is dominant strategy **IC**. Suppose supplier i bids $(\hat{c}_i, \hat{q}_i)$. Then, for all possible misreports $\hat{\mathbf{b}}$ of suppliers other than i, we have

$$\begin{aligned}
& t_i^{**}((\hat{c}_i,\hat{q}_i),\widehat{\mathbf{b}}_{-i}) - \hat{c}_i x_i^*((\hat{c}_i,\hat{q}_i),\widehat{\mathbf{b}}_{-i}) \\
&= \int_{c_i}^{\bar{c}} x_i^*((u,\hat{q}_i),\widehat{\mathbf{b}}_{-i})du \\
&\quad + \int_{\hat{c}_i}^{c_i} x_i^*((u,\hat{q}_i),\widehat{\mathbf{b}}_{-i})du - (c_i-\hat{c}_i)x_i^*((\hat{c}_i,\hat{q}_i),\widehat{\mathbf{b}}_{-i}), \\
&\leq \int_{c_i}^{\bar{c}} x_i^*((u,\hat{q}_i),\widehat{\mathbf{b}}_{-i})du,, \qquad (2.19) \\
&\leq \int_{c_i}^{\bar{c}} x_i^*((u,q_i),\widehat{\mathbf{b}}_{-i})du,, \qquad (2.20) \\
&= t_i^{**}(\mathbf{b}_i,\widehat{\mathbf{b}}_{-i}) - c_i x_i^*(\mathbf{b}_i,\widehat{\mathbf{b}}_{-i}),
\end{aligned}$$

where inequality (2.19) follows from the fact that $x_i^*((c_i,q_i),\mathbf{b}_{-i})$ is non-increasing in c_i for all $(q_i,\mathbf{b}_{-i})$ (see Lemma 2.2 (a)) and inequality (2.20) is a consequence of the fact that $x_i^*((c_i,q_i),\mathbf{b}_{-i})$ is non-decreasing in q_i for all $(c_i,\mathbf{b}_{-i})$ (see Lemma 2.2 (b)). Thus, truth-telling forms a dominant strategy equilibrium. ■

2.3.2 Optimal Mechanism in the General Case

In this section, we consider the case when Assumption 2.2 does not hold, i.e. the distribution of the cost and capacity does not satisfy regularity.

The optimal allocation rule is given by the solution to following optimal control problem

$$\begin{aligned}
\max_{\mathbf{x}(\mathbf{b}),\bar{\rho}(\mathbf{q})} \quad & \mathbb{E}_{\mathbf{b}}\left[R\Big(\sum_{i=1}^n x_i(\mathbf{b})\Big) - \sum_{i=1}^n H_i(c_i,q_i)x_i(\mathbf{b}) + \bar{\rho}_i(q_i)\right] \\
s.t \quad & 0 \leq x_i(c_i,q_i) \leq q_i \quad \forall i,q_i,c_i \\
& \hat{c}_i \geq c_i \Rightarrow X_i(\hat{c}_i,q_i) \leq X_i(c_i,q_i) \quad \forall q_i,c_i,\hat{c}_i,i \qquad (2.21) \\
& \hat{q}_i \geq q_i \Rightarrow \int_{c_i}^{\bar{c}} (X_i(z,q_i) - X_i(z,\hat{q}_i))dz \leq \bar{\rho}_i(\hat{q}_i) - \bar{\rho}_i(q_i) \quad \forall c_i,q_i,\hat{q}_i,i \\
& 0 \leq \bar{\rho}_i(q_i) \quad \forall q_i,i
\end{aligned}$$

This problem is a very large scale stochastic program and is, typically, very hard to solve numerically. We characterize the solution, under a condition weaker than regularity, which we call *semi-regularity*.

We adapt the standard one dimensional ironing procedure (Myerson [see, e.g. 53]) to our problem which has a two-dimensional type space. Let $L(c_i, q_i)$ denote the cumulative density along the cost dimension, i.e.

$$L_i(c_i, q_i) = \int_{\underline{c}}^{c_i} f_i(u, q_i) du$$

Since the density $f_i(c_i, q_i)$ is assumed to be strictly positive, $L_i(c_i, q_i)$ is increasing in c_i, and hence, invertible in the c_i coordinate. Let

$$K_i(p_i, q_i) = \int_{\underline{c}}^{c_i} H_i(u, q_i) f_i(u, q_i) dt$$

where $c_i = L_i(\cdot, q_i)^{-1}(p_i)$. Let $\hat{K}_i$ denote the convex envelop of K_i along p_i, i.e.

$$\hat{K}_i(p_i, q_i) = \inf \Big\{ \lambda K_i(a, q_i) + (1-\lambda) K_i(b, q_i) | a, b \in [0, L_i(\bar{c}, q_i)], \\ \lambda \in [0,1], \lambda a + (1-\lambda) b = p_i \Big\}.$$

Define ironed-out virtual cost function $\widehat{H}_i(c_i, q_i)$ by setting it to

$$\widehat{H}_i(c_i, q_i) = \frac{\partial \hat{K}_i}{\partial p}(p, q) \Big|_{p_i = L_i(c_i, q_i), q_i}$$

wherever the partial derivative is defined and extending it to $[\underline{c}, \bar{c}]$ by right continuity.

Lemma 2.3. *The function K_i, the convex envelop $\hat{K}_i$ and the ironed-out virtual costs $\widehat{H}(c_i, q_i)$ satisfy the following properties.*

1. *$\widehat{H}_i(c_i, q_i)$ is continuous and nondecreasing in c_i for all fixed q_i.*

2. *$\hat{K}_i(0, q_i) = K_i(0, q_i)$, $\hat{K}_i(L_i(\bar{c}, q_i), q_i) = K_i(L_i(\bar{c}, q_i), q_i)$,*

3. *For all q_i and p_i, $\hat{K}_i(p_i, q_i) \leq K_i(p_i, q_i)$.*

4. *Whenever $\hat{K}_i(p_i, q_i) < K_i(p_i, q_i)$, there is an interval (a_i, b_i) containing p_i such that $\frac{\partial}{\partial p}\hat{K}(p, q_i) = c$, a constant, for all $p \in (a_i, b_i)$. Thus, $\widehat{H}_i(c_i, q_i)$ is constant with $c_i \in L_i(\cdot, q_i)^{-1}((a_i, b_i))$.*

See Rockafeller [61] for the proofs of these assertions. Now, we are ready to state our weaker regularity assumption.

Assumption 2.3 (Semi-Regularity)**.** *For all $i = 1, 2, \cdots, n$, the ironed out virtual marginal production cost, $\widehat{H}_i(c_i, q_i)$ is non-increasing in q_i.*

From Lemma 2.3 (a) above, it follows that the semi-regularity implies the usual regularity of $\widehat{H}_i$, i.e. $\widehat{H}_i$ satisfies Assumption 2.2. Theorem 2.4 shows that if we use this ironed out virtual cost function in the buyer's profit function instead of the original virtual cost and then pointwise maximize to find the optimal allocation relaxing the monotonicity constraints on the optimal allocation and the side payments $\bar{\rho}_i$, then the resulting mechanism is incentive compatible with $\bar{\rho}_i = 0$ and revenue maximizing.

Theorem 2.4. *Suppose Assumption 2.3 holds. Let $\mathbf{x}^{\mathrm{I}} : \mathbf{B} \to \mathbb{R}^n_+$ denote any solution of the pointwise optimization problem*

$$\max_{\mathbf{0} \le \mathbf{x} \le \mathbf{q}} \left\{ R\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} x_i \widehat{H}_i(c_i, q_i) \right\}.$$

Set the transfer payment function

$$t_i^{\mathrm{I}}(\mathbf{b}) = c_i x_i^{\mathrm{I}}(\mathbf{b}) + \int_{c_i}^{\bar{c}} x_i^{\mathrm{I}}((u, q_i), \mathbf{b}_{-i}) du. \tag{2.22}$$

Then $(\mathbf{x}^{\mathrm{I}}, \mathbf{t}^{\mathrm{I}})$ is a revenue maximizing, dominant strategy incentive compatible and individually rational procurement mechanism.

Proof: Let $\mathbf{x}$ be any **IC** allocation and let $\boldsymbol{\rho}$ denote the corresponding offered surplus. Define

$$\widehat{\Pi}(\mathbf{x}, \bar{\boldsymbol{\rho}}) \equiv \mathbb{E}_b \left[R\left(\sum_{i=1}^{n} x_i(b)\right) - \sum_{i=1}^{n} x_i(b) \widehat{H}_i(c_i, q_i) - \sum_{i=1}^{n} \bar{\rho}_i(q_i) \right],$$

i.e. $\widehat{\Pi}(\mathbf{x}, \bar{\boldsymbol{\rho}})$ denotes buyer profit when the virtual costs $H_i(c_i, q_i)$ are replaced by the ironed-out virtual costs $\widehat{H}_i(c_i, q_i)$. Then

$$\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}}) - \widehat{\Pi}(\mathbf{x}, \bar{\boldsymbol{\rho}}) = \int_{\underline{q}}^{\bar{q}} \left[\int_{\underline{c}}^{\bar{c}} \left(\widehat{H}_i(c_i, q_i) - H_i(c_i, q_i) \right) X_i(c_i, q_i) f_i(c_i, q_i) dc_i \right] dq_i$$

The inner integral

$$\begin{aligned} &\int_{\underline{c}}^{\bar{c}} \left(\widehat{H}_i(c_i, q_i) - H_i(c_i, q_i) \right) X_i(c_i, q_i) f_i(c_i, q_i) dc_i \\ &= \left(\hat{K}_i(c_i, t) - K_i(c_i, t) \right) \Big|_0^{L_i(c_i, q_i)} - \int_{\underline{c}}^{\bar{c}} \left(\hat{K}_i(c_i, q_i) - K_i(c_i, q_i) \right) f_i(c_i, q_i) d_{c_i} \left[X_i(c_i, q_i) \right], \\ &= - \int_{\underline{c}}^{\bar{c}} \left(\hat{K}_i(L(c_i, q_i), q_i) - K_i(L(c_i, q_i), q_i) \right) f_i(c_i, q_i) \partial_{c_i} X_i(c_i, q_i), \qquad (2.23) \\ &\leq 0, \qquad (2.24) \end{aligned}$$

where (2.23) follows from Lemma 2.3 (b), and (2.24) follows from Lemma 2.3 (c) and the fact that $\partial_{c_i} X_i(c_i, q_i) \leq 0$ for any **IC** allocation rule. Thus, we have the $\widehat{\Pi}(\mathbf{x}, \boldsymbol{\rho}) \geq \Pi(\mathbf{x}, \bar{\boldsymbol{\rho}})$.

A proof technique identical to the one used to prove Theorem 2.3 establishes that $(\mathbf{x}^{\mathrm{I}}, \mathbf{t}^{\mathrm{I}})$ is an dominant strategy **IC** and **IR** procurement mechanism that maximizes the ironed-out buyer profit $\widehat{\Pi}$. Note that the corresponding offered surplus $\bar{\rho}^{\mathrm{I}}(q) \equiv 0$.

Suppose $\hat{K}_i(L(c_i, q_i), q_i) < K_i(L(c_i, q_i), q_i)$. Then Lemma 2.3 (d) implies that $H_i(c_i, q_i)$ is a constant for some neighborhood around c_i, i.e. $\partial_{c_i} X_i(c_i, q_i) = 0$ in some neighborhood of c_i. Consequently, the inequality (2.24) is an equality when $\mathbf{x} = \mathbf{x}^I$, i.e. $\widehat{\Pi}(\mathbf{x}^{\mathrm{I}}, \boldsymbol{\rho}^{\mathrm{I}}) = \Pi(\mathbf{x}^{\mathrm{I}}, \boldsymbol{\rho}^{\mathrm{I}})$. This establishes the result. ■

Theorem 2.4 characterizes the revenue maximizing direct mechanism when the virtual costs $H_i(c_i, q_i)$ satisfy semi-regularity, or equivalently, when the ironed-out virtual costs $\widehat{H}_i(c_i, q_i)$ satisfy regularity. When semi-regularity does not hold, the optimal direct mechanism can still be computed by numerically solving the stochastic program (2.21). Our numerical experiments lend support to the following conjecture.

Conjecture 2.1. *A revenue maximizing procurement mechanism has the following properties.*

1. *The side payments* $\bar{\boldsymbol{\rho}} \equiv \mathbf{0}$.
2. *There exist completely ironed-out virtual cost functions* $\tilde{H}_i$ *such that the corresponding point-wise solution* $\widetilde{\mathbf{x}} = \text{argmax}\left\{R(\sum_i \tilde{x}_i(\mathbf{b})) - \sum_{i=1}^n \tilde{H}_i(c_i, q_i)\tilde{x}_i(\mathbf{b}) : \mathbf{0} \leq \tilde{\mathbf{x}} \leq \mathbf{q}\right\}$ *is the revenue maximizing* **IC** *allocation rule.*
3. *The ironing procedure and the completely ironed-out virtual costs* $\tilde{H}_i(c_i, q_i)$ *depend on the* revenue function R, *in addition to the joint prior.*

Rochet and Chone [59] present a general approach for multidimensional screening but in a model where the agents have both sided incentives i.e. they are allowed to mis-report their type both below and above it's true value.

2.3.3 Low-bid Implementation of the Optimal Auction

In this section, we assume that all the suppliers are identical, i.e. $F_i(c, q) = F(c, q)$, and that the distribution $F(c, q)$ satisfies Assumption 2.1 and Assumption 2.2. From (2.16) it follows that the expected transfer payment

$$T_i^*(c_i, q_i) = c_i X_i^*(c_i, q_i) + \int_{c_i}^{\bar{c}} X_i^*(u, q_i) du$$

Note that $T_i^*(c_i, q_i) = 0$, whenever $X_i^*(c_i, q_i) = 0$. Define a new point-wise transfer payment $\widetilde{\mathbf{t}}$ as follows.

$$\tilde{t}_i(\mathbf{b}) = \left(c_i + \frac{\int_{c_i}^{\bar{c}} X_i^*(t, q_i) dt}{X_i^*(c_i, q_i)} \right) x_i^*(c_i, q_i) \tag{2.25}$$

Then $\mathbb{E}_{(c_{-i},q_{-i})}\left[t_i(c,q)\right] = T_i^*(c_i,q_i)$, therefore, $(\mathbf{x}^*, \tilde{\mathbf{t}})$ is Bayesian **IC** and **IR**. We use the transfer function $\widetilde{\mathbf{t}}$ to compute the bidding strategies in a "low bid" implementation of the direct mechanism. The "get-your-bid" auction proceeds as follows:

1. Supplier i bids the capacity $\hat{q}_i \leq q_i$, she is willing to provide and the marginal payment p_i she is willing to accept.

2. The buyer's action are as follows:

   (a) Solve for the true marginal cost c_i by setting[1]

   $$p_i = \phi(c_i, \hat{q}_i) = c_i + \frac{\int_{c_i}^{\bar{c}} X_i^*(t, \hat{q}_i)dt}{X_i^*(z, \hat{q}_i)}.$$

   (b) Aggregates these bids and forms the virtual procurement cost function $\tilde{c}$ by setting

   $$\tilde{c}(q) = \sum_{j=1}^{i-1} \hat{q}_{[j]} h_{[j]} + \Big(q - \sum_{j=1}^{i-1} \hat{q}_{[j]}\Big) h_{[i]} \tag{2.26}$$

   for $\sum_{j=1}^{i-1} \hat{q}_{[j]} \leq q \leq \sum_{j=1}^{i} \hat{q}_{[j]}$, where as before $h_{[i]}$ denotes the i-th lowest virtual marginal cost and $\hat{q}_{[i]}$ is the capacity bid of the corresponding supplier.

   (c) Solve for the quantity $\tilde{q} = \text{argmax}\{R(q) - \tilde{c}(q)\}$. Set the allocation

   $$\tilde{x}_{[i]} = \begin{cases} \hat{q}_{[i]}, & \sum_{j=1}^{i} \hat{q}_{[j]} \leq \tilde{q}, \\ \tilde{q} - \sum_{j=1}^{i-1} \hat{q}_{[j]}, & \sum_{j=1}^{i-1} \hat{q}_{[j]} \leq q \leq \sum_{j=1}^{i} \hat{q}_{[j]}, \\ 0 & \text{otherwise.} \end{cases}$$

3. Supplier i produces $\tilde{x}_i$ and receives $\tilde{p}_i \tilde{x}_i$.

When all the suppliers are identical, the expected allocation function $X_i^*(c,q)$ is independent of the supplier index i. We will, therefore, drop the index.

[1] We assume that $\phi(c_i, q_i)$ is strictly increasing in c_i, for all q_i. This would be true, for example when the virtual costs H_i are strictly increasing in c_i. Note that previously, we had been working with allocations that were non-decreasing.

Theorem 2.5. *The bidding strategy*

$$\begin{aligned} \tilde{q}(c,q) &= q, \\ \tilde{p}(c,q) &= \phi(c,q) \equiv c + \frac{\int_c^{\bar{c}} X^*(t,q)dt}{X^*(c,q)}, \end{aligned}$$

is a symmetric Bayesian Nash equilibrium for the "get-your-bid" procurement mechanism.

Proof: Comparing (2.14) and (2.26), it is clear that, in equilibrium, $\mathbf{x}^*(\mathbf{b}) = \widetilde{\mathbf{x}}(\mathbf{p},\mathbf{q})$. Assume that all suppliers except supplier i use the bidding the proposed bidding strategy. Then the expected profit $\pi_i(\hat{p}_i, \hat{q}_i)$ of supplier i is given by

$$\begin{aligned} \pi_i(\hat{p}_i, \hat{q}_i) &= (\hat{p}_i - c_i)\tilde{X}_i(\hat{p}_i, \hat{q}_i) \\ &= (\hat{p}_i - c_i)X_i^*(\hat{c}_i, \hat{q}_i), \end{aligned}$$

where $\hat{c}_i$ given by the solution of the equation $\hat{p}_i = \phi(c, \hat{q}_i)$. Thus, we have that

$$\begin{aligned} \pi_i(\hat{p}_i, \hat{q}_i) &\leq (\hat{p}_i - c_i)X_i^*(\hat{c}_i, q_i), && (2.27) \\ &= \int_{\hat{c}_i}^{\bar{c}} X^*(u,q_i)du - (\hat{c}_i - c_i)X^*(c_i,q_i), \\ &= \int_{c_i}^{\bar{c}} X^*(u,q_i)du + \int_{\hat{c}_i}^{\bar{c}} X^*(u,q_i)du(\hat{c}_i - c_i)X^*(c_i,q_i), \\ &\leq \int_{c_i}^{\bar{c}} X^*(u,q_i)du = \pi(\tilde{p}(c_i,q_i),\tilde{q}(q_i)), && (2.28) \end{aligned}$$

where (2.27) and (2.28) follows from, respectively, Lemma 2.2 (b) and (a). Thus, it is optimal for supplier i to bid according to the proposed strategy. ■

2.3.4 Corollaries

Since the point-wise profit in (2.12) depends on the capacity q_i only through the conditional distribution $F_i(c_i|q_i)$ of the cost c_i given capacity q_i, the following result is immediate.

Corollary 2.1. *Suppose the marginal cost c_i and capacity q_i are independently distributed. Then the optimal allocation rule (and the corresponding transfer function) is insensitive to the capacity distribution.*

Contrasting this result with the "get-your-bid" implementation in the last section, we find that although the optimal auction mechanism is insensitive to the capacity, the supplier bidding strategies may depend on the capacity distribution.

The following result characterizes the buyers profit function when the suppliers' capacity is common knowledge.

Corollary 2.2. *Suppose suppliers' capacity is common knowledge. Then the buyers expected profit under any feasible,* **IC** *and* **IR** *allocation rule* $\mathbf{x}$ *is given by*

$$\Pi(\mathbf{x}) = \mathbb{E}_c \left[R\left(\sum_{i=1}^{n} x_i(c) \right) - \sum_{i=1}^{n} x_i(c) \left(c_i + \frac{F_i(c_i)}{f_i(c_i)} \right) \right] \tag{2.29}$$

Suppose the buyer wishes to procure a fixed quantity Q from the suppliers. Since a given realization of the capacity vector $\mathbf{q}$ can be insufficient for the needs to the buyer, i.e. $\sum_{i=1}^{n} q_i < Q$, we have to allow for the possibility of an exogenous procurement source. We assume that the buyer is able to procure an unlimited quantity at a marginal cost $c_0 > \bar{c}$. Let $EC(Q)$ denote the expected cost of procuring quantity Q by any optimal mechanism.

Corollary 2.3 (Fixed Quantity Auction). *Suppose Assumption 2.1 and 2.2 hold. Then*

$$EC(Q) = \mathbb{E}_{(c,q)} \left\{ \begin{array}{ll} \min & \sum_{i=1}^{n} x_i(c,q) q_i H_i(c_i, q_i) + q_0 c_0 \\ s.t. & \sum_{i=1}^{n} x_i q_i + q_0 = Q \\ & 0 \leq \mathbf{x} \leq q \end{array} \right\}. \tag{2.30}$$

Results in this chapter can be adapted to other principle-agent mechanism design settings. Consider monopoly pricing with capacitated consumers. Suppose the monopolist seller with a strictly convex production cost $c(x)$ faces a continuum of customers with utility of the form

$$u(x,t;\theta,q) = \begin{cases} \theta x - t, & x \leq q, \\ -\infty, & x > q, \end{cases}$$

where (θ, q) is the private information of the consumers. The form of the utility function $u(x,t;\theta,q)$ prevents the customer from overbidding its capacity to consume. This is necessary for the seller to be able to check individual rationality. As always the type distribution $F : [\underline{\theta}, \bar{\theta}] \times [\underline{q}, \bar{q}] \rightarrow \mathbb{R}_{++}$ is common knowledge.

Corollary 2.4. *Suppose the distribution $F(\theta, q)$ satisfies the regularity assumption that $\nu(\theta, q) = \theta - \frac{1-F(\theta|q)}{f(\theta|q)}$ is separately non-decreasing in both θ and q. Then the following holds for monopoly pricing with capacitated buyers.*

1. *The seller profit $\Pi(x)$ for any feasible,* **IC** *allocation rule x, the seller expected profit is of the form*

$$\Pi(x) = \mathbb{E}_{(\theta,q)}\left[\Big(\theta - \frac{1-F(\theta|q)}{f(\theta|q)}\Big)x(\theta,q) - c(x(\theta,q))\right]$$

2. *An optimal direct mechanism is given by the allocation rule*

$$x^*(\theta,q) = \underset{0 \leq x \leq q}{\operatorname{argmax}} \left[\Big(\theta - \frac{1-F(\theta|q)}{f(\theta|q)}\Big)x - c(x)\right]$$

   and transfer payment

$$t^*(\theta,q) = \int_{\underline{\theta}}^{\theta} x^*(t,q)dt$$

Since the type space is two-dimensional, the optimal direct mechanism can be implement by a posted tariff only if the parameter θ and the capacity q are independently distributed.

All our results in this section easily extend to nonlinear convex production cost $c_i(\theta, x)$, $\theta \in [\underline{\theta}, \bar{\theta}]$, that are super-linear, i.e. $\frac{\partial^2 c_i}{\partial\theta\partial x} > 0$. In this case, the virtual production cost $H_i(\theta_i, x)$ is given by

$$H_i(\theta_i, x) = c_i(\theta_i, q_i, x) + c_{i\theta}(\theta_i, x)\frac{F_i(\theta_i|q_i)}{f_i(\theta_i|q_i)}.$$

2.4 Multiple Component Model

We next consider a model for procuring variable quantities of multiple products (or components). Consider a situation where a buyer (typically a large manufacturing corporation or large retail chain like Walmart) wants to purchase m different product types. The expected revenue of the buyer as a function of quantity procured is denoted by $R : \mathbb{R}^m \rightarrow \mathbb{R}$. The production cost incurred by supplier i in producing the bundle $\mathbf{x} \in \mathbb{R}^m_+$ is given by $c^i(\theta_i, \mathbf{x})$, where $\theta_i \in [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$ is privately known. The restriction that the production cost is parameterized by a single parameter θ is required to keep the problem tractable. However, we allow general cost functions in order to model asymmetry in production costs.

Assumption 2.4. *The revenue function R and the production cost function $c^i(\theta_i, \mathbf{x})$, $i = 1, \ldots, n$, satisfy the following regularity conditions.*

i) The revenue function R is twice continuously differentiable, concave and supermodular[2]*. In addition, for all $j = 1, \ldots, m$, $\frac{\partial R(\mathbf{x})}{\partial x_j} \rightarrow 0$ as $x_j \rightarrow \infty$ and $\frac{\partial R(\mathbf{x})}{\partial x_j} \rightarrow \infty$ as $x_j \rightarrow 0$.*

ii) The production cost function $c^i(\theta, \mathbf{x})$ satisfies $c^i_\theta(\theta, \mathbf{x})$[3] *$> 0$, $\nabla_x c^i(\theta, \mathbf{x}) \geq \mathbf{0}$, $\nabla_x c^i_\theta(\theta, \mathbf{x}) \geq \mathbf{0}$, $c^i_{\theta\theta}(\theta, \mathbf{x}) \geq 0$, $\mathbf{c}^i_{xx}(\theta, \mathbf{x}) \succeq 0$ and $\mathbf{C}^i_{\theta xx}(\theta, \mathbf{x}) \succeq 0$, where $\mathbf{M} \succeq \mathbf{0}$ denotes that the matrix $\mathbf{M}$ is symmetric positive semidefinite.*

Since the revenue function $R(\mathbf{x})$ is supermodular, we are implicitly assuming that the bundle $\mathbf{x}$ has complementary products.

We assume that each supplier has fixed capacity for each product. Let $\mathbf{a}_i$ denote the capacity vector of supplier i. Then an allocation $\mathbf{x}_i$ to supplier i is feasible

[2] *Recall that a twice continuously differentiable function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is supermodular if the cross derivatives $\frac{\partial 2R}{x_i x_j} \geq 0$ $i \neq j$.*

[3] *$c^i_\theta(\theta, x)$ denote the partial derivative of the production costs of supplier i with respect to θ at (θ, x) and so on.*

only if $\mathbf{x}_i \leq \mathbf{a}_i$. The capacity $\mathbf{a}_i$ exogenously given so that supplier do not have an option of acquiring additional capacity. The capacity vectors $\mathbf{a}_i \in \mathbf{A}$ where $\mathbf{A}$ is a convex subset of $\mathbb{R}^m_+$. The prior distribution $f_i : [\underline{\theta}, \bar{\theta}] \times \mathbf{A} \to \mathbb{R}_{++}$ of the supplier i type $(\theta_i, \mathbf{a}_i)$ is common knowledge. The following regularity assumption on the prior distribution is analogous to Assumption 2.1 in the previous section.

Assumption 2.5. *The prior density f_i has full support. Thus, the conditional density $f_i(\theta|\mathbf{a})$ has full support for all $\mathbf{a} \in \mathbf{A} \subset \mathbb{R}^m_+$.*

We take the direct mechanism approach to construct the optimal direct mechanism. Thus, supplier i's bid $\hat{\mathbf{b}}_i$ is of the form $\hat{\mathbf{b}}_i = (\hat{\theta}_i, \hat{\mathbf{a}}_i) \in [\underline{\theta}, \bar{\theta}] \times \mathbf{A}$. As in the case in the previous section, we assume overbidding capacity results in a heavy penalty. Therefore, the supplier i capacity bid $\hat{\mathbf{a}}_i$ must satisfy $\hat{\mathbf{a}}_i \leq \mathbf{a}_i$, where the inequality is interpreted component-wise. Let $\hat{\mathbf{b}} = [\hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_n]$ and let $\mathbf{B} = \left([\underline{\theta}, \bar{\theta}] \times \mathbf{A}\right)^n$ denote the type-space of all the suppliers.

1. an allocation rule $\mathbf{x} : \mathbf{B} \mapsto \mathbb{R}^{m \times n}$ that for each bid $\hat{\mathbf{b}} \in \mathbf{B}$ specifies the bundle $\mathbf{x}_i \in \mathbb{R}^m$ to be ordered from each of the suppliers, and

2. a transfer payment function $\mathbf{t} : \mathbf{B} \to \mathbb{R}^n$ that maps each bid vector $\hat{\mathbf{b}}$ to the monetary transfer from the buyer to the suppliers.

The buyer seek an allocation function $\mathbf{x}$ and a transfer function $\mathbf{t}$ that maximizes the ex-ante expected profit

$$\Pi(\mathbf{x}, \mathbf{t}) \equiv \mathbb{E}_{\mathbf{b}} \left[R\left(\sum_{i=1}^n \mathbf{x}_i(\mathbf{b}) \right) - \sum_{i=1}^n t_i(\mathbf{b}) \right]$$

subject to **IC**, and **IR**, and feasibility, i.e. $\mathbf{x}_i(\mathbf{b}) \leq \mathbf{a}_i$. The following result is the analog of Lemma 2.1 for multi-component markets.

Lemma 2.4. *All multi-component procurement auctions satisfy the following.*

1. *A feasible allocation rule* $\mathbf{x} : \mathbf{B} \to \mathbb{R}^n_+$ *is* **IC** *if, and only if, for all* $i = 1, \ldots, n$,

$$C^i(\theta_i, \mathbf{a}_i) \equiv \mathbb{E}_{\mathbf{b}_{-i}}\, c^i_\theta(\theta_i, \mathbf{x}(\mathbf{b}))$$

is non-increasing in θ_i.

2. *A mechanism* $(\mathbf{x}, \mathbf{t})$ *is* **IC** *and* **IR** *if, and only if, the allocation rule* $\mathbf{x}$ *satisfies (a) and the offered surplus* $\rho_i(\theta_i, \mathbf{a}_i)$ *is of the form*

$$\rho_i(\theta_i, \mathbf{a}_i) = \bar{\rho}_i(\mathbf{a}_i) + \int_{\theta_i}^{\bar{\theta}} C^i_\theta(z, \mathbf{a}_i) dz \tag{2.31}$$

with $\rho_i(\theta_i, \mathbf{a}_i)$ *non-negative and non-decreasing in* a_{ij} *for all* $\mathbf{a}_{-j}$ *and* θ.

The proof of this result is very similar to that of Lemma 2.1 and is, therefore, left to the reader to reconstruct.

The fact that the cost parameter θ is one dimensional is crucial for Lemma 2.4 to hold. All allocation rule which are incentive compatible with respect to the cost parameter θ_i can be made incentive compatible with respect to the capacity simply by constructing appropriate side payments $\bar{\rho}_i$ which are independent of θ_i. The following analog of Theorem 2.1 can be easily established using Lemma 2.4.

Theorem 2.6. *Suppose Assumption 2.5 holds. Then the buyers profit corresponding to any allocation rule* $\mathbf{x}$ *that satisfies* **IC** *and* **IR** *is given by*

$$\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}}) = \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^{n} \mathbf{x}_i(\mathbf{b})\right) - \sum_{i=1}^{n} H^i(\theta_i, \mathbf{a}_i, \mathbf{x}_i) - \sum_{i=1}^{n} \bar{\rho}_i(\mathbf{a}_i)\right]$$

where $\bar{\rho}_i$ *are* θ_i*-independent side payments, and*

$$H^i(\theta_i, \mathbf{a}_i, \mathbf{x}_i) \equiv c^i(\theta, \mathbf{x}) + c^i_\theta(\theta, \mathbf{x}) \frac{F_i(\theta|\mathbf{a})}{f_i(\theta|\mathbf{a})} \tag{2.32}$$

are virtual production costs.

Thus, the mechanism design problem reduces to the following optimization problem.

$$\begin{array}{lll} \max_{\mathbf{x}(\theta,\mathbf{a}),\bar{\rho}_i(\mathbf{a}_i)} & \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^n \mathbf{x}_i(\mathbf{b})\right) - \sum_{i=1}^n H^i(\theta_i,\mathbf{a}_i,\mathbf{x}_i) - \sum_{i=1}^n \bar{\rho}_i(\mathbf{a}_i)\right], & \\ \text{s.t.} & \mathbf{x}_i \le \mathbf{a}_i, & \text{for all } i = 1,\ldots,n, \\ & C^i_\theta(\theta_1,\mathbf{a}) \le C^i_\theta(\theta_2,\mathbf{a}) & \text{for all } \theta_1 \ge \theta_2, \theta_1,\theta_2 \in [\underline{\theta},\bar{\theta}], \mathbf{a}\in\mathbf{A}, i = 1,\ldots,n \\ & \bar{\rho}_i(\mathbf{a}_1) \ge \bar{\rho}_i(\mathbf{a}_2) & \mathbf{a}_1 \ge \mathbf{a}_2, \mathbf{a}_1,\mathbf{a}_2 \in \mathbf{A}, i = 1,\ldots,n. \end{array}$$

Next, we discuss conditions under which the point-wise allocation rule

$$\mathbf{x}^*(\mathbf{b}) \in \operatorname*{argmax}_{0\le \mathbf{x}_i \le \mathbf{a}_i} \left\{ R\left(\sum_{i=1}^n \mathbf{x}_i(\theta,a)\right) - \sum_{i=1}^n H_i(\theta_i,\mathbf{a}_i,\mathbf{x}_i)\right\} \tag{2.33}$$

is the revenue maximizing procurement mechanism. Recall that in the context of the single component model, in order for the point-wise optimal allocation rule to be the revenue maximizing allocation rule, one needed to ensure that $X_i^*(c_i,q_i)$ is non-increasing in c_i and non-decreasing in q_i. We need a similar condition here.

Assumption 2.6. *The virtual cost function $H^i(\theta_i,\mathbf{a}_i,\mathbf{x}_i)$ defined in (2.32) satisfies the following regularity conditions.*

1. *For all $\mathbf{b}_i = (\theta_i,\mathbf{a}_i)$, the virtual cost function $H^i(\theta_i,\mathbf{a}_i,\mathbf{x})$ is sub-modular in $\mathbf{x}$.*
2. *The gradient $\nabla_x H^i(\theta,\mathbf{a},\mathbf{x})$ satisfies the following:*
   (a) $\nabla_x H^i(\theta_2,\mathbf{a},\mathbf{x}) \ge \nabla_x H^i(\theta_1,\mathbf{a},\mathbf{x})$ *for all* $\theta_2 \ge \theta_1$.
   (b) $\nabla_x H^i(\theta,(a_j,\mathbf{a}_{-j}),\mathbf{x}) \le \nabla_x H^i(\theta,(a'_j,\mathbf{a}_{-j}),\mathbf{x})$ *for all* $a_j \ge a'_j$.

Note that we are implicitly assuming that the virtual cost function $H^i(\mathbf{b},\mathbf{x})$ is differentiable in $\mathbf{x}$. Assumption 2.6 (i) is satisfied when production cost are separable, i.e. $\frac{\partial^2 c_i(\theta,x)}{\partial x_i \partial x_j} = 0$. If the capacity vector $\mathbf{a}_i$ and the cost parameter θ_i are independently distributed, then Assumption 2.6 (ii) is satisfied when conditional density satisfy $\frac{F_i(\theta|a)}{f_i(\theta|a)} = \frac{F_i(\theta)}{f_i(\theta)}$ is monotone in θ and $\nabla_x c^i_{\theta\theta} \ge 0$.

Theorem 2.7. *Suppose Assumptions 2.4, 2.5 and 2.6 hold and revenue function is linear*[4]. *Define transfer payments*

$$t_i^*(\mathbf{b}) = c^i(\theta_i, \mathbf{x}_i^*) + \int_{\theta_i}^{\bar{\theta}} c_\theta^i(u, \mathbf{x}_i^*)du, \qquad i = 1, \ldots, n. \tag{2.34}$$

Then $(\mathbf{x}^*, \mathbf{t}^*)$ *is a revenue maximizing multi-component procurement auction under which bidding truthfully forms a dominant strategy equilibrium.*

Proof: From the proof technique used to establish the single-component result in Theorem 2.3, it follows that all we need to show is that

1. $c_\theta^i\big(\theta_i, \mathbf{x}^*((\theta_i, \mathbf{a}_i), \mathbf{b}_{-i})\big)$ is non-increasing in θ_i for all $\mathbf{a}_i$ and $\mathbf{b}_{-i}$, and

2. $c_\theta^i\big(\theta_i, \mathbf{x}^*((\theta_i, \mathbf{a}_i), \mathbf{b}_{-i})\big)$ is non-decreasing in a_{ij} for all θ_i, a_{ik}, $k \neq j$, and $\mathbf{b}_{-i}$.

We use the following result.

Theorem 2.8 (Milgrom and Shannon [50], Theorem 4′)**.** *Let* $f : \mathbf{X} \times \mathbf{T} \to \mathbb{R}$*, where* $\mathbf{X}$ *is a lattice and* $\mathbf{T}$ *is a partially ordered set. If* $\mathbf{S} : \mathbf{T} \to \mathbf{2}^{\mathbf{X}}$ *is nondecreasing and if* f *is quasi-supermodular in* $\mathbf{x}$ *and satisfies the single crossing property in* $(\mathbf{x}, \mathbf{t})$*, the every selection* $\mathbf{x}^*(\mathbf{t})$ *from* $\operatorname{argmax}_{\mathbf{x} \in \mathbf{S}(\mathbf{t})} f(\mathbf{x}, \mathbf{t})$ *is monotone nondecreasing in* $\mathbf{t}$*.*

Define

$$\begin{aligned}
\mathbf{T} &= \Big([-\bar{\theta}, -\underline{\theta}] \times \mathbf{A}\Big)^n, \\
\mathbf{X} &= \Big(\mathbb{R}^m\Big)^n \\
\mathbf{S}(t) &= \{\mathbf{x} \mid \mathbf{0} \leq \mathbf{x}_i \leq \mathbf{a}_i, \forall i = 1, \ldots, n\}, \\
f(\mathbf{x}, \mathbf{b}) &= R\Big(\sum_{i=1}^n \mathbf{x}_i\Big) - \sum_{i=1}^n H^i(\mathbf{b}_i, \mathbf{x}_i),
\end{aligned}$$

The ordering for both the lattice $\mathbf{X} \subset \prod_{i=1}^n \mathbb{R}^{m+1}$ and the partially-ordered set $\mathbf{T} \subseteq \prod_{i=1}^n \mathbb{R}^n$ is the usual product ordering.

[4] *This theorem holds either* $n = 1$ *or* $R(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + b$ *for some* $\mathbf{a} \gg 0, b \geq 0$ *or* $n = 2$ *and* $R(\mathbf{x}) = \sum_{j=1}^m R_j(x_j)$*.*

It is clear that $\mathbf{S}(t)$ is "box-shaped" and is increasing in each of the component of a_{ij} in the usual strong order on $\mathbf{R}^m$. By Assumption 2.4 (i), Assumption 2.6 (i) and the linearity of R, it follows that $f(\mathbf{x}, \mathbf{b})$ is quasi-supermodular in $\mathbf{x}$ for all $\mathbf{b} \in \mathbf{T}$.

Let $\mathbf{x}^2 \geq \mathbf{x}^1$ and $\mathbf{b}^2 \geq \mathbf{b}^1$. Then

$$\begin{aligned}
f(\mathbf{x}^2, \mathbf{b}^2) - f(\mathbf{x}^1, \mathbf{b}^2) &= \left(R\Big(\sum_{i=1}^n \mathbf{x}_i^2\Big) - R\Big(\sum_{i=1}^n \mathbf{x}_i^1\Big)\right) - \sum_{i=1}^n \left(H^i(\mathbf{b}_i^2, \mathbf{x}_i^2) - H^i(\mathbf{b}_i^2, \mathbf{x}_i^1)\right), \\
&= \left(R\Big(\sum_{i=1}^n \mathbf{x}_i^2\Big) - R\Big(\sum_{i=1}^n \mathbf{x}_i^1\Big)\right) \\
&\quad - \sum_{i=1}^n \int_0^1 \nabla_x H^i(\mathbf{x}_i^1 + (\mathbf{x}_i^2 - \mathbf{x}_i^1)u, \mathbf{b}^2)^T (\mathbf{x}_i^2 - \mathbf{x}_i^1) du, \\
&\geq \left(R\Big(\sum_{i=1}^n \mathbf{x}_i^2\Big) - R\Big(\sum_{i=1}^n \mathbf{x}_i^1\Big)\right) \\
&\quad - \sum_{i=1}^n \int_0^1 \nabla_x H^i(\mathbf{x}_i^1 + (\mathbf{x}_i^2 - \mathbf{x}_i^1)u, \mathbf{b}^1)^T (\mathbf{x}_i^2 - \mathbf{x}_i^1) du, \qquad (2.35) \\
&= f(\mathbf{x}^2, \mathbf{b}^1) - f(\mathbf{x}^1, \mathbf{b}^1),
\end{aligned}$$

where (2.35) follows from Assumption 2.6 (ii). Thus, it follows that $f(\mathbf{x}, \mathbf{b})$ satisfies the single-crossing property in $(\mathbf{x}, \mathbf{b})$.

Consequently, Theorem 2.8 implies that $\mathbf{x}^*(\mathbf{b})$ is monotonically non-decreasing in $\mathbf{b}$. Since the order of the θ-component of the type vector was reversed (see the definition of $\mathbf{T}$), we have that $\mathbf{x}^*(\mathbf{b})$ is non-increasing in θ_i and non-decreasing in the capacity $\mathbf{a}_i$. The result now follows from the regularity conditions in Assumption 2.4-(ii). ■

2.4.1 Auctioning Multi-period Supply Chain Contract

We apply the multi-product model to the setting where a manufacturer wants to procure units of a single good over multiple periods. We assume full commitment and do not allow for renegotiation. This assumption is reasonable in high-value industries where the end-consumers, even though elastic in their consumption at the time of contract negotiation, would expect agreed-upon deliveries at the time of actual consumption.

Suppose the time horizon of the contracts consists of w periods. The production vector $\mathbf{x}_i$ of supplier i is of the form $\mathbf{x}_i = (x_{i1}, \ldots, x_{iw}) \in \mathbb{R}_+^w$, where x_{it} denotes the production in period t. The full-commitment assumption implies that the vector $\mathbf{x}_i$ can be treated as a bundle of T different products with production cost $c^i(\theta_i, \mathbf{x}_i)$. The capacity constraints for the supplier i are modeled as before: $\mathbf{x}_i \leq \mathbf{a}_i$. This capacity model implicitly assumes that the suppliers' production is limited due to some exogenous effects other than resources.

Suppose the cost function $c^i(\theta_i, \mathbf{x}_i)$, the revenue function $R : \mathbb{R}_+^w \to \mathbb{R}_+$ and the prior distributions $f_i(\theta_i, \mathbf{a}_i)$ satisfy conditions of Theorem 2.7, i.e. we are in the so-called "regular" case. Then the optimal allocation $\mathbf{x}$ is given by the solution of the point-wise optimization problem (2.33) and this allocation is be implemented by the transfer payments $\mathbf{t}$ defined in (2.34).

Suppose the cost function $c^i(\theta_i, \mathbf{x})$ and revenue function $R(\mathbf{x})$ are both separable in $\mathbf{x}$, i.e. the quantities procured and sold in different time periods do not interact. Then point-wise optimization problem is separable. Thus, the reverse auction reduces to m single-component auctions – one for each time period. These auctions can be either be held sequentially at the beginning of each period or all together at time zero.

2.5 Conclusion and Extensions

We presented a procurement mechanism that is able to optimally screen for both privately known capacities and privately known cost information. The results can be easily adapted to other principle-agent mechanism design problems in which agents have a privately known bounds on consumption.

We explicitly characterize the optimal procurement mechanism when the costs and the prior distribution satisfy the so-called "regularity" condition - a form of negative affiliation between capacities and costs (see Assumption 2.2). Under regularity, the private information about capacities works to the advantage of the buyer as it reveals some information about the costs. We show that the optimal mechanism cannot be implemented as uniform price auction, e.g. K^{th} price auction. We also show that the two-dimensional private information forbids a $P-Q$ curve indirect implementation (see, e.g. Deshpande and Schwarz [19]; Chen [13]) of the optimal direct mechanism.

In the absence of regularity, there are complex tradeoffs between incentives to reveal the capacity and the incentives to reveal cost. Consequently, the optimal auction mechanism is a solution to complex stochastic program. In this case, we are only able to characterize the mechanism when a certain "semi-regularity" condition holds.

Our model can be easily extended to multi-product case with complementarities when the private cost information is single dimensional. An application of this model in auctioning multi-period supply contract also showed that a buyer with full commitment who faces the inter-temporal risk of variable capacities over time from the cost efficient supplier can effectively hedge this risk by committing in the beginning to order from different suppliers in different periods.

Some natural extension of this works as follows:

1. Suppliers can ex-post purchase additional capacity q^a at a cost $g(q^a)$. In this case, the optimal supplier surplus can be written as

$$\pi_i(c_i, q_i) = \underset{(\hat{c}_i, \hat{q}_i)}{\operatorname{argmax}} \left\{ T_i(\hat{c}_i, \hat{q}_i) - c_i X_i(\hat{c}_i, \hat{q}) - C_i^q(c_i, q_i) \right\} \tag{2.36}$$

where

$$C_i^q(c_i, q_i) = \mathbb{E}_{(c_{-i}, q_{-i})}\left[\mathbf{1}\left(x_i((\hat{c}_i, c_{-i}), (\hat{q}_i, q_{-i})) > q_i\right) g_i\big(x_i((\hat{c}_i, c_{-i}), (\hat{q}_i, q_{-i})) - q_i\big)\right]$$

Unlike in (2.1), the true capacity q explicitly appears in (2.36). This makes the problem truly 2-dimensional and Lemma 2.1 fails to hold.

2. Multi-product model with multidimensional private information about the product cost: Even the uncapacitated version of this problem remains unsolved.

3. Construct more robust implementation of the optimal direct mechanism: the possibility of using dynamic games of incomplete information which converge the equilibrium of the one shot game.

Chapter 3

Optimal Sponsored Search Auctions

3.1 Introduction

In this chapter we study sponsored search auctions used by the internet search service providers such as Google and Yahoo!. Sponsored search advertising is a major source of revenue for internet search engines. Close to 98% of Google's total revenue of $7.14 billion for the year 2006 came from sponsored search advertisements. It is believed that more than 50% of Yahoo!'s revenue of $6.4 billion in the year 2006 was from sponsored search advertisement.

Sponsored search advertisements work as follows. A user queries a certain *adword*, i.e. a keyword relevant for advertisement, on an online search engine. The search engine returns the links to the most "relevant" webpages and, in addition, displays certain number of relevant sponsored links in certain fixed "slots" on the result page. Every time the user clicks on any of these sponsored links, she is taken to the website of the advertiser sponsoring the link and the search engine receives certain price per click from the advertiser. The average number of clicks per unit time on a given sponsored link is called its "click-through-rate". It is reasonable to expect that, all things being equal, a user is more likely click on the link that is placed in a slot that is easily visible on the page. In any case, the click-through-rate is a function of the slot, and therefore, advertisers have a preference over which slot carries their link and are willing to pay a higher price per click when placed on a more desirable slot. The click-through-rate of a sponsored link is likely to be an increasing function of the exogenous brand values of the advertisers; therefore, search engines prefer allocating more desirable slots to advertisers with higher exogenous brand value. In conclusion, the search engines need a mechanism for allocating slots to advertisers. Since auctions are very effective mechanisms for revenue generation and efficient allocation, they have become the mechanism of choice for assigning sponsored links to advertising slots.

Adwords auctions are dynamic in nature – the advertisers are allowed to change their bids quite frequently. In this chapter, we design and analyze static models for adword auctions. We use the dominant strategy solution concept in order to ensure that the static model adequately approximates dynamic adword auctions.

Our models are far from capturing all the tradeoffs relevant for current business models for adword pricing. In particular, we don't model the influence of budgets, risk averseness, bidder irrationality (or bounded rationality) and diversification across adwords. There is growing literature which focuss on these aspects on adword auctions, some of which we discuss in our literature survey. We use the phrases "sponsored search auctions" and "adword auctions" interchangeably in this chapter - both are common in literature.

This chapter is organized as follows. We discuss the relevant literature in § 3.1.1. In § 3.2 we describe the general adword auction model. In § 3.3 we discuss the adword auction model with slot independent private valuations per click. In § 3.4 we propose and analyze an adword auction model in which the valuation per click are a privately known constant up to privately known constant and zero thereafter - also called the slotted model. In § 3.5 we discuss the results of the numerical study and § 3.6 contains some concluding remarks.

3.1.1 Previous Literature

The online adword auctions have caught the attention of the academic community only recently – after all, sponsored search and the Internet itself is a recent phenomena when compared with the long tradition of research in auction theory. The papers that specifically address adword auctions from the perspective of auction theory include Aggarwal et al. [2], Edelman et al. [21] and Lahaie [42]. Edelman et al. [21] present a comprehensive introduction and history of the adword auctions. Edelman et al. [21] study an adword auction model with slot independent valuation, separable click-though-rate and generalized second price payment rule. They observe that truth-telling is not a dominant strategy for this auction. Aggarwal et al. [2] also study the same model and propose a payment rule under which any *rank based* allocation rule can be made dominant strategy incentive compatible. They also show that if the click-though-rate is separable (in which case the Vickrey-Clark-Groves (VCG) allocation rule is rank based) there exists an ex-post equilibrium which results in same pointwise revenue as the generalized second price auction. This equilibrium is a simple adaptation of the equilibrium in Edelman et al. [21]. We demonstrate that this revenue equivalence follows in a straightforward manner from the fact that the private information in the uniform valuation model is single dimensional. Lahaie [42] characterizes the equilibrium bidding strategies in rank based mechanisms with first price and second price payment schemes in both complete and incomplete information setting.

Varian [67] characterizes the Nash Equilibrium in an adword auction with first price payments. In this model, the value per click only depends on the identity of the bidder and the click through rate only depends on the slot. Varian [67] also reports results of comparing the prices predicted by the Nash equilibrium to empirical prices.

Zhan et al. [73], Feng [25] and Lim and Tang [45] present a Bayesian Nash analysis of a related adword auction model with one dimensional private information. Feng and Zhang [27] study bid price cycling in online auctions. Feng et al. [26] assume truthful bidding and study the revenue performance of alternative rank based mechanisms using simulation. Liu and Chen [46] propose using historical bid as the prior for designing the auctions. Kitts et al. [40] present a simplified analysis of the equilibrium behavior with very few assumptions, focussing on dynamic behavior and empirical analysis of the bid data.

Shapley and Shubik [64] describe an efficient assignment game in which bidders are assigned to objects with each bidder receiving at most one object. See Bikhchandani and Ostroy [9] for a recent survey of mechanisms that yield efficient equilibria for the assignment game. Leonard [44] showed that a specific optimal dual solution of the matching linear program implements the efficiency maximizing allocation in dominant strategy.

The characterization of the incentive compatibility constraint in the slotted model is based on our results in chapter 2 where we analyzed one-sided incentives in a reverse auction model. Vohra and Malakhov [70] also consider a similar but restrictive model. We show in § 3.4 that this restrictive model is not adequate in the setting considered in the slotted model in this chapter (see Example 3.3). Aggarwal et al. [1] propose a *top-down* auction for a slotted model with bidder independent click-through-rate and show that this auction has an envy-free Nash equilibrium with the same allocation and prices as the efficiency maximizing VCG mechanism. We show in § 3.3 that the top-down allocation rule is a special case of the customized rank-based allocation rule.

3.2 Sponsored Search Auction Model

There are n advertisers bidding for $m(\leq n)$ slots on a specific adword. Let c_{ij} denote the click-through-rate when advertiser i is assigned to slot j. For convenience, we will set $c_{i,m+1} = 0$ for all $i = 1, \ldots, n$.

Assumption 3.1. *The click through rates $\{c_{ij}\}$ satisfy the following conditions.*

(i) For all bidders i, the rate c_{ij} is strictly positive and non-increasing in j, i.e. all *bidders rank the slots in the same order.*

(ii) The rates c_{ij}, for all *(i,j) pairs $i = 1, \ldots, n$, $j = 1, \ldots, m$, are known to the auctioneer.*

(iii) Only the rates $(c_{i1}, c_{i2}, \ldots, c_{im})$ are known to bidder i, i.e. each bidder only knows her click-through-rates.

The true expected per-click-value v_{ij} of slot j to advertiser i is private information. We assume *independent private values* (IPV) setting with a commonly known prior distribution function that is continuously differentiable with density $f(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \prod_{i=1}^{n} f_i(\mathbf{v}_i) : \mathbb{R}_+^{m\times n} \mapsto \mathbb{R}_{++}$. Note that even through we use dominant strategy as the solution concept, we still need the prior distribution in order to select the optimal mechanism. We restrict attention to direct mechanisms – the revelation principle guarantees that this does not introduce any loss of generality.

Let $\mathbf{b} \in \mathbb{R}_+^{n\times m}$ denote the bids of the n bidders. An auction mechanism for this problem consists of the following two components.

1. An allocation rule $\mathbf{X} : \mathbb{R}_+^{n\times m} \mapsto \{0,1\}^{n\times m}$ that satisfies

$$\sum_{i=1}^{n} X_{ij}(\mathbf{b}) = 1, \quad j = 1, \ldots, m,$$
$$\sum_{j=1}^{m} X_{ij}(\mathbf{b}) \leq 1, \quad i = 1, \ldots, n.$$

Thus, $\mathbf{X}(\mathbf{b})$ is a matching that matches bidders to slots as a function of the bid $\mathbf{b}$. Henceforth, we denote the set of all possible matchings of n advertisers to m slots by $\mathcal{M}_{nm}$.

2. A payment function $\mathbf{T} : \mathbb{R}_{+}^{n\times m} \mapsto \mathbb{R}^{n}$ that specifies what each of the n bidders pay the auctioneer.

We show below that one can set the payment of the bidder who is not allocated any slot to zero without any loss of generality. Thus, we can define the per click payment t_i of advertiser i as

$$t_i(\mathbf{b}) = \frac{T_i(\mathbf{b})}{\sum_{j=1}^{m} c_{ij} X_{ij}(\mathbf{b})}.$$

For $\mathbf{v} \in \mathbb{R}_{+}^{n\times m}$ and $i = 1, \ldots, n$, let

$$\hat{u}_i(\mathbf{b}, \mathbf{v}; (\mathbf{X}, \mathbf{T}), \mathbf{v}_{-i}) = \sum_{j=1}^{m} \big(c_{ij} v_{ij} - T_i(\mathbf{b}, \mathbf{v}_{-i})\big) X_{ij}(\mathbf{b}, \mathbf{v}_{-i}) \tag{3.1}$$

denote the utility of the advertiser i of type $\mathbf{v}_i$ who bids $\mathbf{b}$. When the mechanism $(\mathbf{X}, \mathbf{T})$ is clear by context, we will write the utility as $\hat{u}_i(\mathbf{b}, \mathbf{v}; \mathbf{v}_{-i})$.

We restrict attention to mechanisms $(\mathbf{X}, \mathbf{T})$ that satisfy the following two properties:

1. Incentive compatibility (**IC**): For all $\mathbf{v} \in \mathbb{R}_{+}^{n\times m}$ and all $i = 1, \ldots, n$,

$$\mathbf{v}_i \in \operatorname*{argmax}_{\mathbf{b} \in \mathbb{R}_{+}^{m}} \big\{\hat{u}_i(\mathbf{b}, \mathbf{v}_i; (\mathbf{X}, \mathbf{T}), \mathbf{v}_{-i})\big\}, \tag{3.2}$$

i.e. truth telling is ex-post dominant.

2. Individual rationality (**IR**): For all $\mathbf{v} \in \mathbb{R}_{+}^{n\times m}$ and all $i = 1, \ldots, n$,

$$\operatorname*{argmax}_{\mathbf{b} \in \mathbb{R}_{+}^{m}} \big\{\hat{u}_i(\mathbf{b}, \mathbf{v}_i; (\mathbf{X}, \mathbf{T}), \mathbf{v}_{-i})\big\} \geq 0, \tag{3.3}$$

i.e. we implicitly assume that the outside alternative is worth zero.

In the remainder of this chapter we use **IC** as a shorthand for dominant strategy incentive compatibility and **IR** as a shorthand for ex-post individual rationality. Let

$$u_i(\mathbf{v}_i; (\mathbf{X}, \mathbf{T}), \mathbf{v}_{-i}) = \max_{\mathbf{b} \in \mathbb{R}^m_+} \{\hat{u}_i(\mathbf{b}, \mathbf{v}_i; (\mathbf{X}, \mathbf{T}), \mathbf{v}_{-i})\} \tag{3.4}$$

denote the maximum attainable utility for advertiser i under the mechanism $(\mathbf{X}, \mathbf{T})$. If the mechanism $(\mathbf{X}, \mathbf{T})$ is **IC** and **IR** then clearly,

$$u_i(\mathbf{v}_i; (\mathbf{X}, \mathbf{T}), \mathbf{v}_{-i}) = \hat{u}_i(\mathbf{v}_i, \mathbf{v}_i; (\mathbf{X}, \mathbf{T}), \mathbf{v}_{-i}) \tag{3.5}$$

Next, we develop an alternative characterization of the **IC** constraint. Fix $\mathbf{v}_{-i}$ and consider the optimization problem of i-th bidder,

$$u_i(\mathbf{v}_i, \mathbf{v}_{-i}) = \max_{\hat{\mathbf{v}}_i \in \mathbb{R}^m_+} \Big\{ \sum_{j=1}^{m} (c_{ij} v_{ij} - \mathbf{T}_i(\hat{\mathbf{v}}_i, \mathbf{v}_{-i})) X_{ij}(\hat{\mathbf{v}}_i, \mathbf{v}_{-i}) \Big\}.$$

Clearly u_i is convex in $\mathbf{v}_i$ since it is a maximum of a collection of linear functions and by envelope conditions its gradient (under truth-telling) is given by

$$\nabla_{\mathbf{v}_i} u_i(\mathbf{v}_i, \mathbf{v}_{-i}) = (c_{i1} X_{i1}(\mathbf{v}_i, \mathbf{v}_{-i}), \ldots, c_{im} X_{im}(\mathbf{v}_i, \mathbf{v}_{-i}))^{\mathrm{T}} \quad a.e.$$

Thus, an incentive compatible allocation rule is always a sub-gradient of some convex function and hence integrable and monotone[1], as was reported by Laffont et al. [41].

Several authors have characterized the set of incentive compatible allocation rule in quasi-linear environments[2] (see Lavi et al. [43]; Vohra and Malakhov [69]; Chung and Ely [15]; Saks and Yu [62]) in terms of the absence of negative 2-cycles (also called weak monotonicity):

$$\sum_{j=1}^{m} c_{ij} \left(v_{ij} X_{ij}(\mathbf{v}) + \tilde{v}_{ij} X_{ij}(\tilde{\mathbf{v}}_i, \mathbf{v}_{-i}) \right) \geq \sum_{j=1}^{m} c_{ij} \left(v_{ij} X_{ij}(\tilde{\mathbf{v}}_i, \mathbf{v}_{-i}) + \tilde{v}_{ij} X_{ij}(\mathbf{v}) \right) \quad \forall \mathbf{v}_i, \tilde{\mathbf{v}}_i, \mathbf{v}_{-i},$$

[1] The function $\mathbf{X} : \mathbb{R}^m_+ \mapsto \mathbb{R}^m$ is monotone if for every $\mathbf{y}, \mathbf{z} \in \mathbb{R}^m_+$, $(\mathbf{y} - \mathbf{z})^{\mathrm{T}}(\mathbf{X}(\mathbf{y}) - \mathbf{X}(\mathbf{z})) \geq 0$

[2] See Appendix A for the definition of quasi-linear utility environments.

i.e. the sum of utility allocated to advertiser i at $\mathbf{v}_i$ and $\tilde{\mathbf{v}}_i$ under truthful bidding is greater than the sum of utility allocated to advertiser i if he lies and bid $\tilde{\mathbf{v}}_i$ at $\mathbf{v}_i$ and $\mathbf{v}_i$ at $\tilde{\mathbf{v}}_i$. For a convex domain this condition implies that the allocation $\mathbf{X}$ is integrable and the transfer payments implementing $\mathbf{X}$ are well defined. Chung and Ely [15] propose a new implementability condition called quasi-efficiency according to which an allocation rule $\mathbf{X}$ is **IC** if, and only if, for all i, θ, there exist functions $g_i : \Theta^{n-1} \times \mathcal{A} \mapsto \mathbb{R}$ such that

$$\mathbf{X}(\theta) = \underset{a \in \mathcal{A}}{\operatorname{argmax}} \left\{ v_i(\theta_i, a) + g_i(\theta_{-i}, a) \right\},$$

where θ is the private information, $\mathcal{A}$ is the set of allocations and $v_i(\theta_i, a)$ is the valuation function. For efficient allocations, the functions g_i is just the sum of utilities of all bidders other than i at their respective type θ_{-i} and allocations a.

One drawback of these characterizations is that they guarantee existence of transfer payments but do not provide any way of computing them. We give a new characterization of **IC** allocation rules directly in terms of discriminatory bidder dependent slot prices.

Lemma 3.1. *An allocation rule* $\mathbf{X} : \mathbb{R}_+^{n \times m} \mapsto \mathcal{M}_{nm}$ *is incentive compatible if and only if for all* $1 \leq i \leq n$ *and* $\mathbf{v}_{-i} \in \mathbb{R}_+^{(n-1) \times m}$*, there exists per-click prices* $\mathbf{p}_i(\mathbf{v}_{-i}) \in (\mathbb{R} \cup \{\infty\})^m$ *and* $p_{i0}(\mathbf{v}_{-i}) \in \mathbb{R} \cup \{\infty\}$ *such that,*

$$X_{ij}(\mathbf{v}) = 1 \Rightarrow c_{ij}(v_{ij} - p_{ij}(\mathbf{v}_{-i})) \geq \max \left\{ \{ c_{ik}(v_{ik} - p_{ik}(\mathbf{v}_{-i})) \}_{k=1}^{m}, -p_{i0}(\mathbf{v}_{-i}) \right\}.$$

Proof: Fix i, $\mathbf{v}_{-i}$ and suppress the dependence on $\mathbf{v}_{-i}$. $\mathbf{X}$ is **IC** iff there exists convex functions (i.e. the indirect utilities) $u_i : \mathbb{R}_+^m \mapsto \mathbb{R}$ such that

$$\nabla u_i(\mathbf{v}_i) = c_{ij}\mathbf{e}_j \text{ for some } j \in \{0, 1, \ldots, m\} \quad a.e.$$

where $\mathbf{e}_j$ is the j-th unit vector, $\mathbf{e}_0 = \mathbf{0}$ and $c_{i0} = 0 \; \forall i$. Since a convex function is absolutely continuous, it follows that u_i is piecewise linear. Furthermore, a piecewise linear function is convex if, and only if, it is the pointwise maximum of each of its pieces; thus,

$$u_i(\mathbf{v}_i) = \max_{0 \le j \le m} \left\{ c_{ij}\mathbf{e}_j{}^{\mathrm{T}}\mathbf{v}_i - c_{ij}p_{ij} \right\} \quad \forall \mathbf{v}_i. \tag{3.6}$$

Define

$$\mathcal{S}_j = \{\mathbf{v} \in \mathbb{R}_+^m | X_{ij}(\mathbf{v}) = 1\}, \; j = 1, \ldots, m, \quad \text{and}$$
$$\mathcal{S}_0 = \{\mathbf{v} \in \mathbb{R}_+^m | X_{ij}(\mathbf{v}) = 0 \; \forall 1 \le j \le m\}.$$

Recall that $u_i(\mathbf{v}_i) = \sum_{j=1}^m (c_{ij}v_{ij} - T_i(\mathbf{v}_i))X_{ij}(\mathbf{v}_i)$. We claim that $T_i(\mathbf{v}_i) = T_{ij}$ for all $\mathbf{v}_i \in \mathcal{S}_j$, i.e. the payment for bidder i does not change with her bid as long as she gets the same slot[3]. Suppose this is not the case and there exists $\mathbf{v}_i^1 \neq \mathbf{v}_i^2 \in \mathcal{S}_j$ such that $T_i(\mathbf{v}_i^1) < T_i(\mathbf{v}_i^2)$. Then the bidder with valuations $\mathbf{v}_i^2$ would lie and bid $\mathbf{v}_i^1$. Thus,

$$u_i(\mathbf{v}_i) = \sum_{j=1}^{m} (c_{ij}v_{ij} - T_{ij})X_{ij}(\mathbf{v}_i) \tag{3.7}$$

Comparing (3.6) and (3.7), and noting that $X_{ij}(\mathbf{v}) = 1$ iff $\nabla u_i = c_{ij}\mathbf{e}_j$, we get $T_{ij} = c_{ij}p_{ij}$ and

$$\mathbf{X}_i(\mathbf{v}_i) \in \underset{0 \le j \le m}{\operatorname{argmax}} \{c_{ij}(v_{ij} - p_{ij}\}$$

where we set $v_{i0} = 0$ for notational ease. This establishes the result. ■

Since $-p_{i0}$ is the surplus of bidder i when she is not assigned any slot, **IR** implies that $p_{i0} \le 0$. For any given **IC** and **IR** mechanism and a fixed $\mathbf{v}_{-i}$, let $\underline{p} = \min_{0 \le j \le m}(c_{ij}p_{ij}) < 0$ then $\tilde{p}_{ij} = p_{ij} - \underline{p}/c_{ij}$ also satisfies **IC**. To see that $\tilde{\mathbf{p}}$ satisfies **IR**, let $j^* \in \operatorname{argmax}_{0 \le j \le m}\{c_{ij}(v_{ij} - p_{ij})\}$. Then

$$\begin{aligned} c_{ij^*}(v_{ij^*} - p_{ij^*}) &\ge c_{ik}(v_{ik} - p_{ik}), \quad \forall k \neq j^* \\ &\ge -c_{ik}p_{ik}, \qquad\qquad \forall k \neq j^* \\ &\ge -\underline{p}; \end{aligned}$$

[3]This claim has been observed in several previous works in more general settings.

implying $c_{ij^*}(v_{ij^*} - \tilde{p}_{ij^*}) \geq 0$. Since the auctioneer would always like to minimize the bidder surplus, henceforth, we assume that all the prices are restricted to be positive and p_{i0} is set to 0. This also verify our initial claim that the bidders who are not allocated any slot do not make any payment without loss of generality.

The above lemma can be interpreted as follows. An allocation rule, **X** is **IC** if, and only if, there exist bidder-dependent slot prices such that bidders self-select the slot allocated to them by **X**. Thus, any deterministic incentively compatible mechanism is uniquely identified by the pricing rules $p_i : \mathbb{R}_+^{(n-1)\times m} \mapsto (\mathbb{R} \cup \{\infty\}^m$.

To further understand the relationship between the characterization in Lemma 3.1 and integrability and monotonicity of the **IC** allocation rule. Fix $\mathbf{v}_{-i}$. Lemma 3.1 implies an allocation rule is **IC** if and only if the following two conditions are satisfied.

(1) If we increase v_{ij}, keeping $\mathbf{v}_{i,-j}$ constant, there exist a threshold value $\underline{v}_{ij}$ such that for all $v_{ij} \leq \underline{v}_{ij}$, advertiser i is not allocated a slot j and for all $v_{ij} > \underline{v}_{ij}$ advertiser i is allocated slot j.

(2) The normal to the hyperplane separating the region in which advertiser i is allocated slot j (i.e. $\mathcal{S}_j$) and the region in which advertiser i allocated slot k (i.e. $\mathcal{S}_k$) is parallel to $(0, \ldots, c_{ij}, \ldots, -c_{ik}, \ldots, 0)$.

Figure 3.1 illustrates the above conditions when there are only two advertising slots. For fixed $\mathbf{v}_{-i}$, there exist $p_{i1} \geq 0$ and $p_{i2} \geq 0$ such that

$$\mathbf{X}_i = \begin{cases} (0,0) & v_{i1} < p_{i1}, v_{i2} < p_{i2} \\ (1,0) & v_{i1} \geq p_1, c_{i2}(v_{i2} - p_{i2}) \leq c_{i1}(v_{i1} - p_{i1}) \\ (0,1) & \text{otherwise} \end{cases}$$

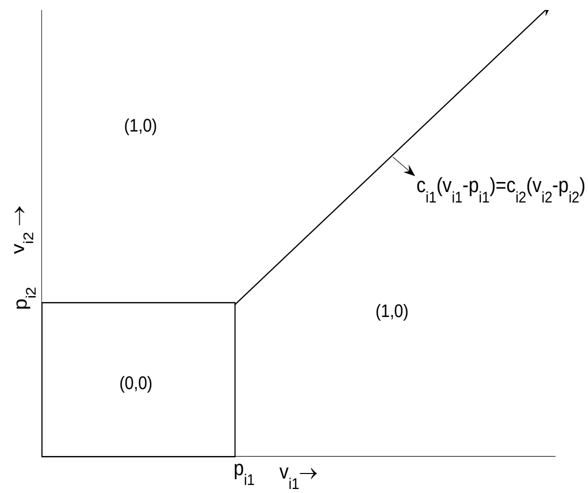


Figure 3.1: Incentive compatible allocation rules

3.2.1 Social Surplus Maximization

It is well known that truth telling can be implemented in dominant strategies by the Vickrey-Clark-Groves (VCG) mechanism (Groves [32]) using the allocation rule $\mathbf{X}^e$ that maximizes the social surplus[4]

$$\Phi(\mathbf{v}, n) = \max_{\mathbf{X} \in \mathcal{M}_{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} v_{ij} X_{ij} \tag{3.8}$$

It is also well known that the linear programming (LP) relaxation of (3.8) obtained by relaxing the constraint $X_{ij} \in \{0, 1\}$ to $X_{ij} \in [0, 1]$ is equivalent to a maximum weighted matching problem on a bipartite graph with unit capacities. For any such network flow problem, there exists an optimal flow that takes values in the set $\{0, 1\}$, i.e. the flow is optimal for (3.8), and can be computed $\mathcal{O}(nm^2)$ time (see, e.g. Ahuja et al. [3]).

Let

$$p_i^e(\mathbf{v}) = \sum_{j=1}^{m} \frac{1}{c_{ij}} \Big\{ c_{ij} v_{ij} - (\Phi(\mathbf{v}, n) - \Phi(\mathbf{v}_{-i}, n-1)) \Big\} X_{ij}^e(\mathbf{v}) \tag{3.9}$$

[4]We use social surplus and efficiency interchangeably in this chapter.

denote the per-click VCG prices. For $j = 1, \ldots, m$, let i_j^* denote the index of the bidder assigned to slot j, i.e. $X^e_{i_j^*,j} = 1$. Leonard [44] established that the vector of slot prices $\nu_j^e = \sum_{i=1}^n p_i^e(\mathbf{v}) X_{ij}^e(\mathbf{v})$, $j = 1, \ldots, m$, are the unique optimal solution of the LP

$$\begin{array}{ll} \min & \sum_{j=1}^m \nu_j \\ \text{s.t.} & \rho_i + \nu_j \geq c_{ij} v_{ij}, \qquad i = 1, \ldots, n, j = 1, \ldots, m, \\ & \rho_{i_j^*} + \nu_j = c_{i_j^* j} v_{i_j^* j}, \quad j = 1, \ldots, m, \\ & \boldsymbol{\rho}, \boldsymbol{\nu} \geq \mathbf{0}. \end{array}$$

Since $\mathbf{X}^e(\mathbf{v})$ is a piecewise constant function of $\mathbf{v}$ and given $\mathbf{X}^e$, $\boldsymbol{\nu}^e(\mathbf{v})$ is a piecewise linear function of $\mathbf{v}$, it follows that the per-click VCG price $p_i^e(\mathbf{v})$ is a piecewise linear function of $\mathbf{v}$.

Let

$$\tilde{\Phi}_j(\mathbf{v}_{-i}) = \max \Big\{ \sum_{k=1, k \neq i}^{n} \sum_{l=1, l \neq j}^{m} c_{kl} v_{kl} x_{kl} : x_{ij} = 1, \mathbf{X} \in \mathcal{M}_{nm} \Big\},$$

i.e. $\tilde{\Phi}_j(\mathbf{v}_{-i})$ denotes the maximum achievable social surplus excluding the bidder i and the slot j. Then the bidder dependent slot prices defined in Lemma 3.1 are given by

$$p_{ij}(\mathbf{v}_{-i}) = \frac{1}{c_{ij}} \Big(\Phi(\mathbf{v}_{-i}, n-1) - \tilde{\Phi}_j(\mathbf{v}_{-i}) \Big). \tag{3.10}$$

It is easy to check that if bidder i is assigned slot j in $\mathbf{X}^e$, the price $p_{ij}(\mathbf{v}_{-i}) = p_i^e(\mathbf{v})$. From (3.10) it is clear that the prices $\mathbf{p}_i(\mathbf{v}_{-i})$ are piece-wise linear functions of $\mathbf{v}_{-i}$. These prices can be efficiently computed by solving the two optimal weighted matching problems.

3.2.2 Revenue Maximization

We first consider the case of $n = 1$. This corresponds to monopoly pricing of stationary advertisement, e.g. lease pricing of slots on the public webpages. It follows from Lemma 3.1 that a revenue maximizing mechanism for a risk-neutral auctioneer is to allow the bidders to self-select slots based on the posted slot prices

$$\mathbf{p}^* \in \underset{\mathbf{p}\in\mathbb{R}^m_+}{\operatorname{argmax}} \sum_{j=1}^{m} p_j c_j \mathbb{E}\Big[\mathbf{1}\Big(j \in \underset{1\le k\le m}{\operatorname{argmax}}\{c_k(v_k - p_k)\}, v_j \ge p_j\Big)\Big], \tag{3.11}$$

where $\mathbb{E}$ denotes the expectation with respect to the prior distribution $f(\mathbf{v})$ of the valuation vector $\mathbf{v} = (v_1, \ldots, v_m)$.

For $n > 1$, the optimal revenue optimizing mechanism is the solution of the stochastic optimization problem

$$\begin{array}{rl} \max & \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^m c_{ij} p_{ij}(\mathbf{v}_{-i}) X_{ij}(\mathbf{v})\right] \\ \text{s.t.} & \mathbf{X}(\mathbf{v}) \in \mathcal{M}_{nm} \quad \text{a.s.} \\ & v_{ij} - p_{ij}(\mathbf{v}) = \max_k \left\{v_{ik} - p_{ik}(\mathbf{v}_{-i})\right\}, \quad \forall \mathbf{v}, i, j \text{ s.t. } x_{ij}(\mathbf{v}) = 1. \end{array} \tag{3.12}$$

The stochastic optimization problem (3.12) is likely to be computationally hard and very sensitive to the prior distribution. See Rochet and Chone [59] for a general treatment of the optimal mechanism design problem with multidimensional type.

3.2.3 Affine Maximizers vs General Pricing Rules

In this section we relate the bidder-dependent per-click slot prices to prices implied by allocation rules that maximize an affine function of the bidder surplus.

In our setting, a mechanism $(\mathbf{X}(\mathbf{v}), \mathbf{T}(\mathbf{v}))$ is called an affine maximizer auction if there exists constants $\{w_{ij}\}$ and $\{r_{ij}\}$ such that $\mathbf{X}(\mathbf{v})$ maximizes

$$\Phi^{w,r}(\mathbf{v}) = \max_{\mathbf{X}\in\mathcal{M}_{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} [w_i c_{ij} v_{ij} - r_{ij}] X_{ij},$$

and the payment $\mathbf{T}_i^{w,r}(\mathbf{v})$ when bidder i is assigned to slot j is given by

$$\mathbf{T}_i^{w,r}(\mathbf{v}) = \frac{1}{w_i}\Big[\Phi^{w,r}(\mathbf{v}_{-i}, n-1) - \Phi^{w,r}(\mathbf{v}) + (w_i c_{ij} v_{ij} + r_{ij})\Big]$$

The constants $\{r_{ij}\}$ can be interpreted as bidder-dependent slot reservation prices. The affine maximizer allocation rule $\mathbf{X}(\mathbf{v})$ also corresponds to an optimal flow in an appropriately defined network flow problem and the payments $\mathbf{T}(\mathbf{v})$ correspond to an appropriately defined minimal dual optimal vector.

The bidder-dependent slot prices $p_{ij}(\mathbf{v}_{-i})$ that implement the affine-maximizer allocation rule are given by

$$p_{ij}(\mathbf{v}_{-i}) = \frac{1}{c_{ij} w_i}\Big[\Phi^{w,r}(\mathbf{v}_{-i}, n-1) - \tilde{\Phi}_j^{w,r}(\mathbf{v})\Big],$$

where

$$\tilde{\Phi}_j(\mathbf{v}_{-i}) = \max\Big\{\sum_{i=1}^{n}\sum_{k=1}^{m}(w_i c_{ik} v_{ik} + r_{ik}) x_{ik} : x_{ij} = 1, \mathbf{X} \in \mathcal{M}_{nm}\Big\}.$$

Since $\Phi^{w,r}(\mathbf{v}_{-i}, n-1)$ and $\tilde{\Phi}_j^{w,r}(\mathbf{v}_{-i})$ are piece-wise linear, the prices $p_{ij}(\mathbf{v}_{-i})$ are piecewise linear functions of $\mathbf{v}$.

Roberts [58] showed that in a quasi-linear preference domain for a large enough type space (in particular, when the type space is $\mathbb{R}^{|\mathcal{A}|}$ where $\mathcal{A}$ is the allocation space) affine maximizers are the *only* dominant strategy implementable allocation rules. Given this result, a natural question that arises is whether there exist **IC** allocation rules in a quasi-linear environment with a given type space that are *not* affine-maximizers. Lavi et al. [43] raise this question for a matching problem which does not satisfy their *conflicting preferences* constraint (see Open Problem 2, page 36). Since Lemma 3.1 does not restrict the bidder-dependent prices $\mathbf{p}_i(\mathbf{v}_{-i})$ to be of a particular form, whereas the bidder-dependent prices corresponding to affine-maximizers are piecewise linear function of $\mathbf{v}$, there is a possibility that affine-maximizers are a strict subset of **IC** allocation rules.

Example 3.1. *Consider an adword auction with two slots and two bidders. Let* $\mathbf{X}(\mathbf{v})$ *denote the allocation rule that allocates slot* 1 *to bidder* 1 *if, and only if,*

$$v_{11} - v_{12} \geq sign(v_{21} - v_{22}) \cdot (\|v_{21} - v_{22}\|)^{1+\gamma}$$

for some $0 < \gamma < 1$ *and to bidder* 2 *otherwise, and assigns slot* 2 *to the unassigned bidder.*

It is easy to check that this allocation rule is **IC** *with prices*

$$\begin{aligned} p_{11}(\mathbf{v}_2) &= \max(0, sign(v_{21}-v_{22})\cdot(\|v_{21}-v_{22}\|)^{1+\gamma}),\\ p_{12}(\mathbf{v}_2) &= \max(0, -p_{11}(\mathbf{v}_2)),\\ p_{21}(\mathbf{v}_1) &= \max(0, sign(v_{11}-v_{12})\cdot(\|v_{11}-v_{12}\|)^{\frac{1}{1+\gamma}}),\\ p_{22}(\mathbf{v}_1) &= \max(0, -p_{21}(\mathbf{v}_1)), \end{aligned}$$

where the last expression follows from the fact that slot 1 *is assigned to bidder* 2 *whenever* $v_{21} - v_{22} > sign(v_{11} - v_{12}) \cdot (\|v_{11} - v_{12}\|)^{\frac{1}{1+\gamma}}$.

The prices $\{p_{ij}(\mathbf{v}_{-i})\}$ *are strictly non-linear functions of* $\mathbf{v}$. *Consequently,* $\mathbf{X}$ *is an* **IC** *allocation rule that is* not *an affine maximizer.*

We conclude this section with the following theorem.

Theorem 3.1. *There exist individually rational and incentive compatible deterministic direct mechanism which are not affine maximizers.*

3.3 Slot Independent (Uniform) Valuations Model

In this section, we consider the special case where the true per-click valuations of all the bidders are slot independent, i.e. $v_{ij} = v_i$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$. Thus, the type-space of the bidders is single-dimensional.

3.3.1 Incentive Compatible Mechanisms

Lemma 3.2. *The following are equivalent characterizations of* **IC** *allocation rules.*

(a) For all i and $\mathbf{v}_{-i}$ the click-through-rate $\sum_{j=1}^{m} c_{ij} X_{ij}(v_i, \mathbf{v}_{-i})$ is non-decreasing in v_i for all v_i.

(b) For all i and $\mathbf{v}_{-i}$ there exist thresholds $0 \le a_{im}(\mathbf{v}_{-i}) \le a_{i,m-1}(\mathbf{v}_{-i}) \le \cdots \le a_{i,1}(\mathbf{v}_{-i}) \le \infty$ such that bidder i is assigned slot j iff $v_i \in (a_{ij}(\mathbf{v}_{-i}), a_{i,j+1}(\mathbf{v}_{-i})]$.

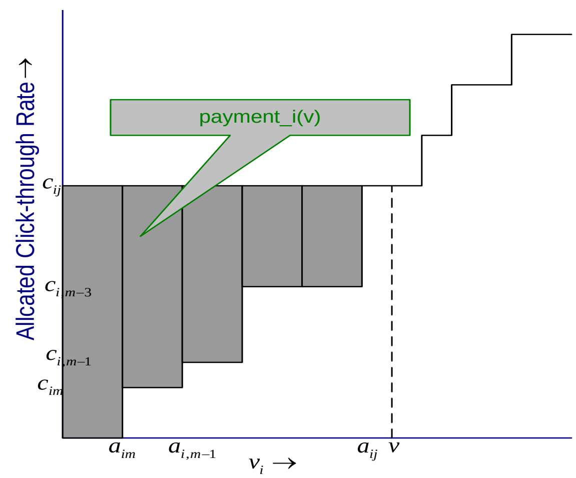


Figure 3.2: Incentive compatibility: allocated click-though-rate as a function of valuation

(c) For all i and $\mathbf{v}_{-i}$*, there exist slot prices* $p_{ij}(\mathbf{v}_{-i})$ *of the form*

$$p_{ij}(\mathbf{v}_{-i}) = \frac{1}{c_{ij}} \sum_{k=j}^{m} \big(a_{ik}(\mathbf{v}_{-i}) - a_{i,k+1}(\mathbf{v}_{-i})\big)(c_{ij} - c_{i,k+1}), \tag{3.13}$$

where $0 \le a_{im}(\mathbf{v}_{-i}) \le a_{i,m-1}(\mathbf{v}_{-i}) \le \cdots \le a_{i,1}(\mathbf{v}_{-i}) \le \infty$ *such that bidders self select the slot assigned to them.*

Proof: Part (a) follows immediately from Holmstrom's Lemma (see, p. 70 in Milgrom [51]).

Since $X_{ij} \in \{0,1\}$ and by Assumption 3.1(i) $c_{ij} \ge c_{i,j+1}$ the total click-through-rate $\sum_{j=1}^{m} c_{ij} X_{ij}(v_i, \mathbf{v}_{-i})$ is non-decreasing with v_i if, and only if, there exist thresholds $0 \le a_{im} \le a_{i,m-1} \le \cdots \le a_{i,1} \le \infty$ such that the allocation rule $\mathbf{X}(\mathbf{v})$ allocates slot j to advertiser i iff $v_i \in (a_{ij}, a_{i,j+1}]$. This is illustrated in Figure 3.2.

It is easy to check that prices of the form (3.13) results in bidder i self-selecting slot j if the valuation $v_i \in (a_{ij}(\mathbf{v}_{-i}), a_{i,j+1}(\mathbf{v}_{-i})]$. Thus, part (b) implies that the prices result in an **IC** allocation rule.

To prove the converse, observe that the payment $T_i(v_i; \mathbf{v}_{-i})$ under any **IC** allocation rule $\mathbf{X}$ must be of the form

$$T_i(v_i; \mathbf{v}_{-i}) = \sum_{j=1}^{m} c_{ij} v_i X_{ij}(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} \Big(\sum_{j=1}^{m} c_{ij} X_{ij}(u, \mathbf{v}_{-i}) \Big) du - u_i(0, \mathbf{v}_{-i}).$$

It is easy to check that $T_i(v_i; \mathbf{v}_{-i}) + u_i(0, \mathbf{v}_{-i})$ is equal to the area of the shaded region in Figure 3.2. Thus per-click price for slot $j = 1, \dots, m$,

$$\begin{aligned} p_{ij}(\mathbf{v}_{-i}) &= \frac{T_i(v_i; \mathbf{v}_{-i}) + u_i(0; \mathbf{v}_{-i})}{c_{ij}}, \\ &= \frac{1}{c_{ij}} \Big\{ (a_{im} - 0)(c_{ij} - 0) + (a_{i,m-1} - a_{im})(c_{ij} - c_{i,m-1}) \\ &\quad + \cdots + (a_{ij} - a_{i,j+1})(c_{ij} - c_{i,j+1}) \Big\} \\ &= \frac{1}{c_{ij}} \sum_{k=j}^{m} (a_{ik} - a_{i,k+1})(c_{ij} - c_{i,k+1}), \end{aligned}$$

where we have set $c_{i0} = 0$ for all $i = 1, \dots, n$. ■

Note that by interchanging the order of integration, i.e. by computing the area of "horizontal" rectangles in Figure 3.2, the prices $p_{ij}(\mathbf{v}_{-i})$ can be alternatively written as

$$p_{ij}(\mathbf{v}_{-i}) = \frac{1}{c_{ij}} \sum_{k=j}^{m} (c_{ik} - c_{i,k+1}) a_{ij}. \tag{3.14}$$

3.3.2 Revenue Maximization

From Myerson [53], it follows that expected revenue of the auctioneer under any dominant strategy incentive compatible allocation rule $\mathbf{X}$ is given by

$$\mathbb{E}\left[\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) X_{ij}(\mathbf{v}) + \sum_{i=1}^{n} u_i(0, \mathbf{v}_{-i}) \right]$$

where $f_i : \mathbb{R}_+ \mapsto \mathbb{R}_{++}$ is the prior density of v_i, $i = 1, \dots, n$. Thus, any two mechanisms (direct or indirect) which at a dominant strategy equilibrium agree on the point wise assignment $\mathbf{X}(\mathbf{v})$ for every $\mathbf{v}$ and on the equilibrium utilities $u_i(0, \mathbf{v}_{-i})$ for all $i, \mathbf{v}_{-i}$ result in identical expected revenues. Aggarwal et al. [2]; Edelman et al. [21] explicitly construct one equilibrium for the *generalized second price* auction which results in same *pointwise* revenue as the truth-telling equilibrium in the direct mechanism.

Let

$$\mathbf{X}^*(\mathbf{v}) \in \underset{\mathbf{X} \in \mathcal{M}_{nm}}{\operatorname{argmax}} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) X_{ij} \right\}.$$

Suppose the virtual valuations per click $\nu_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ are non-decreasing. Then the allocation rule $\mathbf{X}^*$ results in a non-decreasing total-click-through for each of the bidders. Hence, part (a) in Lemma 3.2 implies that $\mathbf{X}^*$ is **IC**. Since the pointwise maximum is an upper bound on any expected revenue maximizing allocation, $(\mathbf{X}^*, \mathbf{T}^*)$ is expected revenue maximizing, dominant strategy **IC**, **IR** allocation rule with per-click prices given by (3.13). When the virtual valuations $\nu(v_i)$ are non-monotonic, the revenue maximizing mechanism can be constructed by first *ironing* (see Myerson [53]) the virtual valuation to obtain a non-decreasing virtual valuations $\tilde{\nu}_i(v_i)$ and then using the construction above.

In the rest of this section, we show how to efficiently compute the payments, or equivalently, thresholds corresponding to the rule $\mathbf{X}^*$. Consider the allocation rules of the form,

$$\mathbf{X}^{\psi}(\mathbf{v}) \in \underset{\mathbf{X} \in \mathcal{M}_{nm}}{\operatorname{argmax}} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \psi_i(v_i) X_{ij} \right\}$$

for any set of $\psi_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$, $\psi_i \in \mathcal{C}[\mathbb{R}]$ and ψ_i non-decreasing. We call $\mathbf{X}^{\psi}$ *monotone maximizer* with respect to the set of monotone transformations $\psi_i, i = 1, \ldots, n$. Given $\mathbf{v}$, $\mathbf{X}^{\psi}(\mathbf{v})$ is the solution to optimal weighted matching problem, and can, therefore, be computed efficiently in $\mathcal{O}(m^2 n)$ time. It is straight forward to observe that $\mathbf{X}^{\psi}$ is incentive compatible and hence has the form presented in Figure 3.2.

As a first step towards computing the slot prices that implement the allocation rule $\mathbf{X}^{\psi}$ we solve the parametric assignment problem

$$\max_{\mathbf{X}\in\mathcal{M}_{nm}} \left\{ \lambda \sum_{j=1}^{m} c_{i_0 j} X_{i_0 j} + \sum_{k=1,k\neq i}^{n} \sum_{j=1}^{m} v_k c_{kj} X_{kj} \right\}, \tag{3.15}$$

where λ is the parameter. It is clear that (3.15) is equivalent to the parametric minimum cost network flow problem

$$\begin{array}{ll} \text{minimize} & -\lambda \sum_{j=1}^{m} c_{i_0 j} X_{i_0 j} - \sum_{k=1,k\neq i}^{n} \sum_{j=1}^{m} v_k c_{kj} X_{kj} \\ \text{subject to} & \sum_{j=1}^{m} X_{ij} - X_{si} = 0 \quad \forall i, \\ & \sum_{i=1}^{n} X_{ij} = 1 \quad \forall j, \\ & \sum_{i=1}^{n} X_{si} = m \end{array} \tag{3.16}$$

on a graph $\mathcal{G}$ defined as follows.

(i) $\mathcal{G}$ has one node for every bidder and slot, and one additional node s.

(ii) $\mathcal{G}$ has unit capacity directed arcs from each bidder $k \neq i_0$ to each slot l with cost $-v_k c_{kl}$ and from s to each bidder with cost zero.

(iii) $\mathcal{G}$ has unit capacity directed arcs from bidder i to each slot j with cost $-\lambda c_{ij}$.

(iv) Each of the slots has unit demand and the node s has a supply of m units.

Lemma 3.3. OPTMATCH *correctly computes the thresholds $a_{i_0 j}$ at which bidder i_0 is assigned to slot $j = 1, \ldots, m$, and the worst case running time of the algorithm is $\mathcal{O}(m^2 n)$.*

Proof: Given a spanning tree structure $(\mathbf{T}, \mathbf{L}, \mathbf{U})$, the node potential π^{λ} corresponding to the parameter value λ is given by $\pi^{\lambda} = \pi^0 + \lambda \pi^1$ with π^0 and π^1 as computed in step 5 and 6. At each optimal spanning tree structure, the non-negativity

Algorithm 3.1 OPTMATCH$(i_0, \mathbf{v}_{-i_0}, c)$

$\mathbf{a} \leftarrow \infty$, $\overline{\lambda} = 0$.
Solve optimal weighted matching problem at $\lambda = 0$.
$(\mathbf{T}, \mathbf{L}, \mathbf{U}) \leftarrow$ optimal spanning tree structure for $\lambda = 0$.
while $\overline{\lambda} < \infty$ **do**
  $\pi^1 \leftarrow$ optimal node potentials at tree $\mathbf{T}$ for $\lambda = 1$ and arc costs equal to zero for all arcs (k, l) with $k \neq i_0$ and $\pi^1(s) \leftarrow 0$.
  $\pi^0 \leftarrow$ optimal node potentials at tree $\mathbf{T}$ for $\lambda = 0$ and $\pi^0(s) \leftarrow 0$
  $\overline{\lambda} \leftarrow \min \left\{ \frac{\pi^0_j - \pi^0_{i_0}}{c_{i_0 j} - \pi^1_j + \pi^1_{i_0}} : (i_0, j) \notin \mathbf{T}, c_{i_0 j} - \pi^1_j + \pi^1_{i_0} \geq 0 \right\}$
  $(i_0, j^*) \leftarrow$ the edge achieving the minimum in the previous step.
  $a_{i_0, j^*} \leftarrow \overline{\lambda}$
  Perform a network-simplex pivot with (i_0, j^*) as the entering arc.
  $(\mathbf{T}, \mathbf{L}, \mathbf{U}) \leftarrow$ optimal spanning tree structure for $\lambda = \overline{\lambda}$.
end while

of reduced costs for each parametric edge (i_0, j) gives a bound on λ as follows:

$$\begin{aligned} & -\lambda c_{i_0,j} - (\pi^0_{i_0} + \lambda \pi^1_{i_0}) + (\pi^0_j + \lambda \pi^1_j) \geq 0 \\ \iff \quad & \lambda (c_{i_0,j} - \pi^1_j + \pi^1_{i_0}) \leq (\pi^0_j - \pi^0_{i_0}) \\ \iff \quad & \begin{cases} \lambda \leq \frac{(\pi^0_j - \pi^0_{i_0})}{(c_{i_0,j} - \pi^1_j + \pi^1_{i_0})}, & c_{i_0,j} - \pi^1_j + \pi^1_{i_0} \geq 0, \\ \lambda \geq \frac{(\pi^0_j - \pi^0_{i_0})}{(c_{i_0,j} - \pi^1_j + \pi^1_{i_0})}, & c_{i_0,j} - \pi^1_j + \pi^1_{i_0} \leq 0. \end{cases} \end{aligned} \tag{3.17}$$

Since $\pi^1_j \geq \pi^1_k$ for each non-parametric edge (k, j), the reduced cost of non-parametric edges, $-v_k c_{kj} - \pi^0_i + \pi^0_j + \lambda(\pi^1_j - \pi^1_k)$ is positive for all values of λ greater than the current λ. Thus, the threshold on $\overline{\lambda}$ up to which the current spanning tree structure remains optimal is equal to the minimum of the upper bounds in (3.17).

After the initial solution, the algorithm performs at most m pivots each costing $\mathcal{O}(mn)$, each returning a threshold at which advertiser i can be assigned slot j. Thus, the overall complexity of OPTMATCH is $\mathcal{O}(m^2 n)$. ■

This proof is adapted from the solution of exercise 11.48 in Ahuja et al. [3].

Next we use OPTMATCH to compute slot prices implementing the monotone maximizer allocation $\mathbf{X}^{\psi}$.

Algorithm 3.2 COMPUTEPRICES

1: $\mathbf{z} \leftarrow (\psi_1(v_1), \ldots, \psi_n(v_n))$, $c_{i,m+1} \leftarrow 0, a_{i,m+1} \leftarrow 0 \quad \forall i$.
2: **for** i=1 to n **do**
3:     $\tilde{\mathbf{a}} \leftarrow \text{OPTMATCH}(i, \mathbf{z}_{-i}, \mathbf{c})$.
4:     **for** $j = 1$ to m **do**
5:         $a_{ij} \leftarrow \psi_i^{-1}(\tilde{a}_{ij})$
6:         **if** $a_{ij} = \infty$ **then**
7:             $a_{ij} \leftarrow \min_{j<k\leq m} a_{ik}$
8:         **end if**
9:     **end for**
10:     **for** j=1 to m **do**
11:         $p_{ij} \leftarrow \frac{1}{c_{ij}} \sum_{k=j}^{m} (a_{ik} - a_{i,k+1})(c_{ij} - c_{i,k+1})$
12:     **end for**
13: **end for**

Lemma 3.4. *Algorithm* COMPUTEPRICES *correctly computes the prices p_{ij} implementing $\mathbf{X}^{\psi}$ in $\mathcal{O}(n^2m^2)$ running time.*

Proof: Without loss of generality, consider the computations for bidder 1. The call to the OPTMATCH returns the thresholds at which bidder 1 get slot j in the virtual valuation space. By the monotonicity[5] of ψ_1, $a_{1p} = \psi_i^{-1}(\tilde{a}_{1p})$ is the corresponding threshold in v-space. Step 6-8 in the algorithm check for the condition that a slot p is never allocated to bidder 1 but a more desirable slot $k < p$ is allocated at $a_{1k} < \infty$ and, in that case, set $a_{1p} = a_{1k}$ (This convention is implied in (3.13)). Step 10-12 use (3.13) to compute the prices p_{1p} given the thresholds.

The dominating computation inside the FOR loops is the call to OPTMATCH, thus giving a $\mathcal{O}(n \cdot m^2 n) = \mathcal{O}(n^2m^2)$ time complexity. ■

The payment corresponding to the revenue maximizing allocation rule $\mathbf{X}^*$ can be computed efficiently using the Lemma 3.4 with $\psi_i(v_i) = \max(\nu_i(v_i), 0)$, where ν_i is the (if necessary) ironed virtual valuations of bidder i.

[5] If $\psi_i(v_i)$ is flat in some interval $\psi_i^{-1}(.)$ is taken to be right continuous at the point of discontinuity.

3.3.3 Rank Based Allocation Rules

Definition 3.1 (Rank based allocation rule)**.** *An allocation function is called rank based if it allocates slot j to the bidder in the j^{th} position under the decreasing order statistics of $\{w_i b_i\}_{i=1}^n$, where $\mathbf{b}$ is the vector of advertiser's bid and $\mathbf{w} \in \mathbb{R}_+^n$ is the allocation function parameter called rank vector.*

Let $\mathbf{X}^w$ denote the rank based allocation rule with ranking vector $\mathbf{w} \in \mathbb{R}_+^n$. Let $\boldsymbol{\gamma} = [w_1 b_1, \ldots, w_n b_n]$. Then under $\mathbf{X}^\mathbf{w}$, the total click-through rate for bidder i is given by

$$\sum_{j=1}^m c_{ij} X_{ij}^w(\mathbf{b}) = \sum_{j=1}^m (c_{ij} - c_{i,j+1}) \mathbf{1}(\gamma_i \geq \gamma_{[j]}^{-i}),$$

where $\gamma_{[j]}^{-i}$ denotes the j-th largest term in the vector $\boldsymbol{\gamma}_{-i}$ and $\mathbf{1}(\cdot)$ denotes the indicator function that takes the value 1 when its argument is true, and zero otherwise. Since γ_i is increasing in b_i and $c_{ij} \geq c_{i,j+1}$, it follows that the total click-through rate $\sum_{j=1}^m c_{ij} X_{ij}^w(\mathbf{b})$ is non-decreasing in b_i. Thus, by Lemma 3.2 part(a), $\mathbf{X}^w$ is incentive compatible.

Lemma 3.5. *The unique per click price $p_{ij}^{\mathbf{w}}$, implementing the rank based allocation rule, $\mathbf{X}^\mathbf{w}$ are given by,*

$$p_{ij}^w(\mathbf{v}_{-i}) = \sum_{k=j}^m \left(\frac{c_{ik} - c_{i,k+1}}{c_{ij}} \right) \frac{\gamma_{[k+1]}}{w_i} \tag{3.18}$$

Proof: Fix $\mathbf{v}_{-i}$. The rank based allocation rule with ranking vector $\mathbf{w}$ allocates slot j to advertiser i iff $\gamma_{[j]} \geq w_i v_i > \gamma_{[j+1]}$. Thus, the thresholds $a_{ij}, j = 1, \ldots, m$ (see Lemma 3.2, part (b)) are equal to $\frac{\gamma_{[j+1]}}{w_i}$. Thus (3.18) follows from the alternative characterization of per click prices in Lemma 3.2, part (c). ■

Aggarwal et al. [2] computes (3.18) using arguments especially tailored for rank based allocations.

The simplicity of the rank based mechanisms make them a very attractive. However, which rank vector $\mathbf{w}$ to use is far from clear! In particular, if the click-though-rate is not separable, i.e. $c_{ij} \neq \phi_i \mu_j$, then both the Google rank vector

($w_i = c_{i1}$) and the Yahoo! rank vector ($\mathbf{w} = \mathbf{1}$) neither maximize efficiency nor maximize revenue. We show that even when the click-through-rate is separable the revenue maximizing rank vector is *not* the same as the efficiency maximizing rank vector and the revenue maximizing mechanism is *not* rank-based!

Example 3.2. *Consider an adword auction of two slots and two bidders with valuations, v_i uniformly distributed on $[0,1]$. Suppose a rank based auction mechanism awards the slot 1 to advertiser 1 if $v_1 \geq \alpha v_2$ and to merchant 2 otherwise. Define $A = c_{11} - c_{12}$ and $B = c_{21} - c_{22}$. Then the prices in (3.18) imply that the expected revenue of the auctioneer,*

$$\Pi(\alpha) = \mathbb{E}_{(v_1,v_2)}\left[A\alpha v_2 \mathbf{1}(v_1 \geq \alpha v_2) + B\frac{v_1}{\alpha}\mathbf{1}(v_2 \geq \frac{v_1}{\alpha})\right]$$

Simplifying, we get

$$\Pi(\alpha) = \begin{cases} A\frac{1}{6\alpha} + B(\frac{1}{2\alpha} - \frac{1}{3\alpha^2}) & \text{if } \alpha \geq 1 \\ A(\frac{\alpha}{2} - \frac{\alpha^2}{3}) + B\frac{\alpha}{6} & \text{if } \alpha \leq 1 \end{cases}$$

Thus the optimum[6]*,*

$$\alpha^* = \begin{cases} \frac{4B}{A+3B} & \text{if } B \geq A \\ \frac{3A+B}{4A} & \text{if } A \geq B \end{cases}$$

Similar calculations shows that the expected social surplus function $\mathbf{S}(\alpha)$ is given by,

$$\mathbf{S}(\alpha) = \begin{cases} \frac{1}{3\alpha}A - \frac{1}{6\alpha^2}B + \frac{1}{2}(c_{21} + c_{12}) & \text{if } \alpha \geq 1 \\ \frac{\alpha}{3}B - \frac{\alpha^2}{6}A + \frac{1}{2}(c_{22} + c_{11}) & \text{if } \alpha < 1 \end{cases}$$

and the efficiency maximizing rank vector $\alpha^e = \frac{B}{A}$. Thus, $\alpha^* \neq \alpha^e$.

[6] *The optimal point can be graphically verified to be unique. For $\alpha \leq 1$, $\Pi(\alpha)$ is concave and for $\alpha \geq 1$, $\Pi(\alpha)$ even though neither convex nor concave, monotonically increases up to a constant* $\min(1, \frac{3A+B}{4A})$ *and decreases thereafter.*

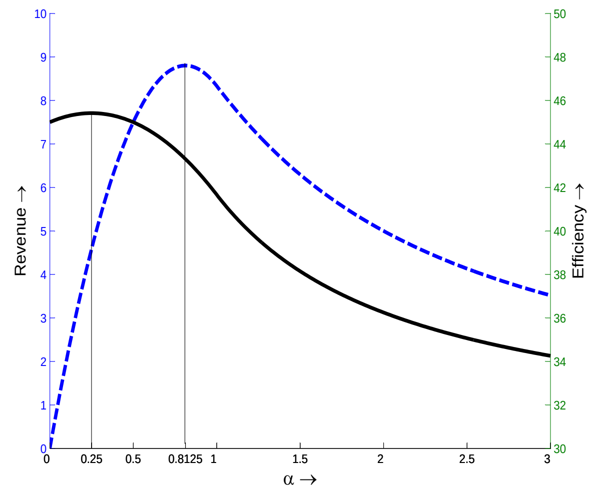


Figure 3.3: Revenue and efficiency as a function of rank weight α in example 3.2

Now, assume that the click-through-rate is separable, $c_{ij} = \phi_i \mu_j$, *for* $\phi > 0$ *and* $\mu_1 \geq \mu_2 > 0$. *Then the revenue maximizing rank vector*

$$\alpha^* = \begin{cases} \frac{4\phi_2}{\phi_1 + 3\phi_2} & \text{if } \phi_2 \geq \phi_1 \\ \frac{3\phi_1 + \phi_2}{4\phi_1} & \text{if } \phi_1 \geq \phi_2 \end{cases}$$

and the efficiency maximizing rank vector $\alpha^e = \frac{\phi_2}{\phi_1}$. *Since* v_i *are uniformly distributed on* $[0, 1]$, *the virtual valuations* $\nu_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} = 2v_i - 1$, *and the revenue maximizing allocation*

$$\mathbf{X}^* \in \underset{\mathbf{X} \in \mathcal{M}_{22}}{\operatorname{argmax}} \sum_{j=1}^{m} \mu_j \sum_{i=1}^{n} \phi_i (2v_i - 1) X_{ij}$$

Thus, $\mathbf{X}^*$ *assigns slot* 1 *to bidder* 1 *if* $v_1 \geq \frac{\phi_2}{\phi_1} v_2 + \frac{\phi_1 - \phi_2}{2\phi_1}$ *and to bidder* 2 *otherwise. This is not a rank based allocation even though the click-through-rate is separable!*

Figure 3.3 plots the efficiency and revenue as a function of α *for* $\mathbf{c} = \begin{bmatrix} 50 & 10 \\ 50 & 40 \end{bmatrix}$. *For this data,* $\alpha^* = 0.8125$ *and* $\alpha^e = 0.25$. *The efficiency maximizing ranking vector being more biased than the revenue maximizing vector because even though bidder 2 generates more value in slot 2, bidder 1 determine the slot price.*

Algorithm 3.3 COMPUTECRPRICES

for $i = 1$ to n **do**
  $\mathbf{S} \leftarrow \phi$, $\mathbf{N}_i \leftarrow \{1, \ldots, n\} \setminus \{i\}$,$j \leftarrow 1$,$c_{i,m+1} \leftarrow 0$, $a_{i,m+1} \leftarrow 0$.
  for $j = 1$ to m **do**
    $(s, I) \leftarrow \max_{k \in \mathbf{N}_i \setminus \mathbf{S}}(\{c_{kj}v_i\})$.
    $a_{ij} \leftarrow \frac{s}{c_{ij}}$, $\mathbf{S} \leftarrow \mathbf{S} \bigcup I$
  end for
  for $j = 1$ to $m - 1$ **do**
    if $a_{ij} < a_{i,j+1}$ **then**
      $a_{i,j+1} \leftarrow a_{ij}$
    end if
  end for
  for j=1 to m **do**
    $p_{ij} \leftarrow \frac{1}{c_{ij}} \sum_{k=j}^{m} (a_{ik} - a_{i,k-1})(c_{ij} - c_{i,k-1})$
  end for
end for

We propose *customized rank based allocation rule* as an improvement over the rank based allocation rule. This rule reduces to the *top dowm* auction proposed in Aggarwal et al. [1] when $c_{ij} \equiv c_j \; \forall i, j$ i.e. bidder independent click-through rates.

Definition 3.2 (Customized Rank Based Allocation)**.** *A customized rank based (***CRB***) allocation rule allocates slot j to the bidder having highest order statistics of $\{c_{ij}b_i\}_{i=1}^{n}$ among those who have not been assigned a slot $< j$.*

Using Lemma 3.2, it is straightforward to deduce that the CRB allocation is incentive compatible. Thus, the prices implementing CRB allocation rule satisfy (3.13). Algorithm COMPUTECRBPRICES computes the prices implementing customized rank based allocation rule in $\mathcal{O}(n^2m)$ time. Establishing correctness of the algorithm is straightforward and is left to the reader. Also, by using the virtual valuations,

$$\mathbf{z} = \left(\left(v_1 - \frac{1 - F_1(v_1)}{f_1(v_1)} \right)^+, \ldots, \left(v_n - \frac{1 - F_n(v_n)}{f_n(v_n)} \right)^+ \right)$$

instead of the valuations $\mathbf{v}$ in Algorithm COMPUTECRBPRICES, the customized rank based allocation rule can approximate revenue maximizing auction as well. In general, the **CRB** allocation rule is likely to out-perform any rank based allocation in terms of both efficiency and revenue generation.

3.3.4 Separable Click-though Rate

Suppose the click-through rate c_{ij} is separable, i.e. $c_{ij} = \phi_i \mu_j$ with $\mu_1 \geq \mu_2 \geq \ldots \mu_m > 0$ and $\boldsymbol{\phi} > \mathbf{0}$. In this setting, the following results immediately follow from our results in the previous section.

1. The solution to the social (virtual) surplus maximization problem is rank based, i.e. the slot j is awarded to the j^{th} order statistics of $\{\phi_i v_i\}_{i=1}^{n}$ (respectively $\{\phi_i \max(0, \nu_i(v_i))\}_{i=1}^{n}$).

2. The price-per-click given by (3.9) implementing the efficiency maximizing allocation are given by

$$p^e_{[i]}(\mathbf{v}) = \frac{1}{\mu_i} \sum_{j=i}^{m} (\mu_j - \mu_{j+1}) \frac{\phi_{[j+1]}}{\phi_{[i]}} v_{[j+1]} \tag{3.19}$$

3. The price-per-click implementing revenue maximizing allocation rule is given by,

$$p^*_{[i]}(\mathbf{v}) = \frac{1}{\mu_i} \sum_{j=i}^{m} (\mu_j - \mu_{j+1}) \nu_{[i]}^{-1} \left(\frac{\phi_{[j+1]}}{\phi_{[i]}} v_{[j+1]} \right) \tag{3.20}$$

   where $\phi_{[k]}$ and $v_{[k]}$ represent the ϕ and bid of advertiser ranked k.

When the click-through-rate is separable, **CRB** allocation rule is the same as the Google allocation rule with rank vector $w_i = c_{i1}$ and is optimal for efficiency maximization.

3.4 Slotted Valuation Model

In this section, we consider the per click valuations of the following form

$$v_{ij} = \begin{cases} v_i, & j \leq k_i \\ 0, & \text{otherwise,} \end{cases}$$

where the tuple (v_i, k_i) is the private information (i.e. type) of the bidder i.

The efficiency maximization problem is given by

$$\begin{array}{rl} \max_{\mathbf{X} \in \mathcal{M}_{nm}} & \sum_{i=1}^{n} \sum_{j=1}^{m} v_i c_{ij} X_{ij} \\ \text{subject to} & X_{ij} = 0, \quad k_i < j \leq m, 1 \leq i \leq n, \end{array} \tag{3.21}$$

Since a bid $\hat{k}_i > k_i$ is ex-post observable, the bidders are restricted to bid $\hat{k}_i \leq k_i$. Since advertisers have one-sided incentives about k_i, this is not a standard mechanism design problem. However, the specific structure of the problem, namely that for all i, $\mathbf{v} \in \mathbb{R}_+^n$ and $\mathbf{k}_{-i}$, the social surplus is non-decreasing in k_i, ensures incentive compatibility with respect to k_i at the VCG payments. When k_i are common knowledge, the solution presented in § 3.3 can be directly applied by pointwise maximizing the virtual surplus.

The following Lemma characterizes the set of all incentive compatible allocation rules in the slotted valuation model. The proof is a simple adaptation of the proof of Lemma 2.1 in Chapter 2 and Lemma 3.2 in this chapter.

Lemma 3.6. *The following are equivalent characterizations of* **IC** *allocation rules.*

(a) The total click-through rate

$$\sum_{j=1}^{k_i} c_{ij} X_{ij}\big((b, \mathbf{k}_i), (\mathbf{v}_{-i}, \mathbf{k}_{-i})\big)$$

for each bidder i is a non-decreasing function of b for all $(\mathbf{v}_{-i}, \mathbf{k})$.

(b) For all $i, \mathbf{v}_{-i}$ *and* $\mathbf{k}$*, there exists thresholds* $0 \leq a_{i,m} \leq a_{i,m-1} \leq \cdots \leq a_{i,k_i} \leq \infty$ *such that bidder* i *is allocated slot* j *iff* $v_i \in (a_{ij}, a_{i,j-1}]$.

(c) The utility of each bidder i *under allocation rule* $\mathbf{X}$ *is of the form*

$$u_i(v_i, k_i; \mathbf{v}_{-i}, \mathbf{k}_{-i}) = \overline{u}_i(k_i, (\mathbf{v}_{-i}, \mathbf{k}_{-i})) + \int_0^{v_i} \Big(\sum_{j=1}^{m} c_{ij} X_{ij}((u, k_i), (\mathbf{v}_{-i}, \mathbf{k}_{-i})) \Big) du.$$

where $\overline{u}_i(k_i, (\mathbf{v}_{-i}, \mathbf{k}_{-i}))$ *are non-negative and nondecreasing functions of* k_i *appropriately chosen to ensure that* $u_i(v_i, k_i; \mathbf{v}_{-i}, \mathbf{k}_{-i})$ *is non-decreasing in* k_i *for all* $(\mathbf{v}, \mathbf{k}_{-i})$.

Lemma 3.6 part (a) and (c) implies that any feasible mechanism that is **IC** with respect to the valuation bid can be made **IC** with respect to the slot bid using independent side payments $\overline{u}_i(k_i, (\mathbf{v}_{-i}, \mathbf{k}_{-i}))$ so that the total surplus of each bidder is increasing with k_i. In the rest of this section, the term *nomimal surplus* will denote the term

$$\int_0^{v_i} \Big(\sum_{j=1}^{m} c_{ij} X_{ij}((u, k_i), (\mathbf{v}_{-i}, \mathbf{k}_{-i})) \Big) du.$$

Lemma 3.6, together a change in the order of integration (see Theorem 2.1 in chapter 2), implies that the revenue maximization problem is equivalent to choosing a feasible allocation rule $\mathbf{X}$ with $X_{ij}(\mathbf{v}) = 0$ for all $k_i < j \leq m, 1 \leq i \leq n$ and a set of side payments $\overline{u}_i$ that maximize

$$\max_{\mathbf{X}, \overline{\mathbf{u}}} \mathbb{E}_{(\mathbf{v},\mathbf{k})} \sum_{i=1}^{n} \left[\sum_{j=1}^{m} c_{ij} \nu_i(v_i, k_i) X_{ij}(\mathbf{v}) - \overline{u}_i(k_i, \mathbf{v}_{-i}) \right]$$

subject to

$$\sum_{j=1}^{m} c_{ij} X_{ij}(v_i, \mathbf{v}_{-i}) \text{ is non-decreasing in } v_i \text{ for all } i, \mathbf{v}_{-i}, \mathbf{k}, \text{ and}$$

$$\overline{u}_i(k_i, \mathbf{v}_{-i}) + \int_0^{v_i} \Big(\sum_{j=1}^{m} c_{ij} X_{ij}((u, k_i), (\mathbf{v}_{-i}, \mathbf{k}_{-i})) \Big) du \text{ is non-decreasing with}$$

k_i for all $i, \mathbf{v}, \mathbf{k}_{-i}$, where $\nu_i(v_i, k_i)$ the virtual valuation per click is defined as follows

$$\nu_i(v_i, k_i) = v_i - \frac{1 - F_i(v_i|k_i)}{f_i(v_i|k_i)}$$

Since the social surplus depends on k_i, there exist bid profiles at which the total click through rate assigned to some bidder i is not monotone in k_i, and, consequently, the nominal surplus,

$$u_i^0(\mathbf{v}, \mathbf{k}) \triangleq \int_0^{v_i} \Big(\sum_{j=1}^{m} c_{ij} X_{ij}((u, k_i), (\mathbf{v}_{-i}, \mathbf{k}_{-i})) \Big) du \tag{3.22}$$

is not monotone in k_i. Therefor, no regularity conditions on the prior distribution $f_i(v_i, k_i)$ can guarantee that the side payments $\overline{u}_i$ can be set equal to zero without loss of generality. Example 3.3 presents a concrete example to demonstrate this.

Example 3.3. *Let* $\mathbf{c} = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$*, and the realized* $k_1 = k_2 = 2$*. Suppose* v_i *and* k_i *are independent, thus the virtual valuations are defined by the marginal distribution of the valuations and in particular, does not depend on* k_i*. The nominal surplus, given by (3.22), of bidder 2*

$$u_2 = \begin{cases} 1(v_2 - 0) & \text{if } \hat{k}_2 = 2 \text{ and } v_2 < a_{21} = \nu_2^{-1}(\frac{3}{2}\nu_1(v_1)) \\ 3(v_2 - \hat{a}_{21}) & \text{if } \hat{k}_2 = 1 \text{ and } v_2 > \hat{a}_{21} = \nu_2^{-1}(\nu_1(v_1)) \end{cases}$$

Thus, if (v_1, v_2) *are such that* $a_{21} > v_2 > \frac{3}{2}\hat{a}_{21}$*, bidder 2 strictly prefers to bids* $\hat{k}_2 = 1$*. Suppose the realized* $\nu_1(v_1) = \frac{4}{17}$*, then bidder 2 strictly prefers to bid* $\hat{k}_2 = 1$ *if* $\frac{3}{2}\nu_2^{-1}\left(\frac{4}{17}\right) < v_2 < \nu_2^{-1}\left(\frac{6}{17}\right)$*.*

Suppose the CDF of v_2 *is given by*

$$F_2(x) = 3x\mathbf{1}_{\{(0,\frac{1}{4})\}}(x) + \left[\frac{3}{4} + \frac{1}{4}\left(1 - e^{-\frac{1}{12}(x - \frac{1}{4})}\right)\right]\mathbf{1}_{\{[\frac{1}{4},\infty)\}}(x)$$

i.e. a $\left(\frac{3}{4}, \frac{1}{4}\right)$ *mixture of a uniform distribution on* $\left(0, \frac{1}{4}\right)$ *and an exponential distribution with rate* $\frac{1}{12}$ *on* $[\frac{1}{4}, \infty)$ *respectively. Then the corresponding virtual valuation function is*

$$\nu_2(x) = \left(2x - \frac{1}{3}\right)\mathbf{1}_{\{(0,\frac{1}{4})\}}(x) + \left(x - \frac{1}{12}\right)\mathbf{1}_{\{[\frac{1}{4},\infty)\}}(x)$$

so that $\hat{a}_{21} = \nu_2^{-1}(\frac{4}{17}) = \frac{29}{102}$ *and* $a_{21} = \nu_2^{-1}(\frac{6}{17}) = \frac{89}{204}$. *Thus for all* $v_2 \in \left(\frac{29}{68}, \frac{89}{204}\right)$ *bidder* 2 *strictly prefers to bid* $\hat{k}_2 = 1$.

Consequently pointwise optimal solution with side payments $\overline{u}$ *set to zero does not induce truth-telling and, hence, is sub-optimal in this example.*

Vohra and Malakhov [70] only considers the **IC** mechanisms with $\overline{u} = 0$. The above example shows that this constraint might eliminate a number of reasonable allocation rules.

Lemma 3.6 part(b) and a simple adaptation of Lemma 3.4 gives the following result.

Lemma 3.7. *Given any set of* $\psi_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$, $\psi_i \in \mathcal{C}[\mathbb{R}]$ *and* ψ_i *non-decreasing, the nominal surplus,*

$$\int_0^{v_i} \Big(\sum_{j=1}^{m} c_{ij} X_{ij}^{\psi}((u, k_i), (\mathbf{v}_{-i}, \mathbf{k}_{-i})) \Big) du$$
$$= \sum_{j=1}^{m} \left[c_{ij} v_i - \sum_{k=j}^{m} (a_{ik} - a_{i,k+1})(c_{i,j} - c_{i,k+1}) \right] X_{ij}^{\psi}(\mathbf{v}, \mathbf{k})$$

can be computed in $\mathcal{O}(m^2 n^2)$ *for any solution to (3.21) with* v_i *replaced by* $\psi_i(v_i)$.

The proof of this lemma is left to the reader.

Next, we discuss two heuristic mechanisms for maximizing revenue. The first heuristic mechanism is defined as follows. If there exists intervals over which the virtual valuations $\nu_i(v_i, k_i) = v_i - \frac{1 - F_i(v_i|k_i)}{f_i(v_i|k_i)}$ is decreasing in v_i, use the ironing procedure detailed in Section 2.3.2 to compute ironed out virtual valuation $\hat{\nu}_i(v_i, k_i)$ that is non-decreasing in v_i. Set the allocation rule

$$\mathbf{X}(\mathbf{v}, \mathbf{k}) \in \operatorname{argmax} \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \hat{\nu}_i(v_i, k_i) X_{ij}(\mathbf{v}). \tag{3.23}$$

Since $\hat{\nu}_i$ is non-decreasing in v_i, $\mathbf{X}(\mathbf{v}, \mathbf{k})$ is **IC**. Set the transfer payment

$$T_i(\mathbf{v}, \mathbf{k}) = \sum_{j=1}^{k_i} c_{ij} v_i X_{ij}(\mathbf{v}, \mathbf{k}) - u_i^0(\mathbf{v}, \mathbf{k}) - \overline{u}_i(\mathbf{v}, k_i) \tag{3.24}$$

where

$$\overline{u}_i(\mathbf{v},\mathbf{k}) = \max_{\hat{k}_i \leq k_i} \left[u_i^0((v_i,\hat{k}_i),\mathbf{t}_{-i}) - u_i^0((v_i,k_i),\mathbf{t}_{-i}) \right] \tag{3.25}$$

The above side payments ensure that the surplus (i.e. the nominal surplus plus the side payments) of each bidder i is non-decreasing in k_i. Since side payments are identically zero in the VCG mechanism, it follows that the side payment above would be identically zero if the virtual valuation are replaced by valuation. Since the nominal surplus $u_i^0((v_i,k_i)$ can be computed in $\mathcal{O}(m^2n^2)$ (see Lemma 3.7) for all bidders at a given bid profile it follows that this mechanism can be implemented in $\mathcal{O}(m^3n^2)$ time by calling Algorithm 3.2 at most m times.

Second mechanism is **CRB** as described in § 3.3 applied to the slotted valuation model, i.e., for $j = 1,\ldots,m$ allocate the slot j to a bidder having highest order statistics of $\{c_{ij}\nu_i(v_i,k_i)\}_{i=1}^n$ among those who have not been assigned a slot number less than j and who have $k_i \leq j$.

Assume that for all i, v_i, the virtual valuation function ν_i is non-decreasing in v_i and k_i. Since, for all i, $\mathbf{v}_{-i}$ and $\mathbf{k}$, the click-through-rate of the slot allocated to advertiser i is non-decreasing in v_i, this mechanism is **IC**. Also note that for all i, $\mathbf{v}$, $\mathbf{k}_{-i}$, the click-through-rate of the slot allocated to advertiser i is non-decreasing in k_i, which implies that the nominal surplus, $u_i^0(\mathbf{v},\mathbf{k})$ is non-decreasing in k_i. Thus, we can set the side payments $\overline{u}_i$ to be identically zero for this allocation rule, i.e. under **CRB**, the advertisers does not earn any information rent due to the k_i dimension of the bid. Moreover, algorithm COMPUTECRBPRICES can be used k_i times for each i to compute the prices that implement the slotted **CRB** mechanism in $\mathcal{O}(m^3n)$ time. The following lemma summarize the above results.

Lemma 3.8. *If the virtual valuations $\nu_i(v_i,k_i), i = 1,\ldots,n$ are non-decreasing in v_i and k_i then X^*_{CRB} is incentive compatible with unique minimal prices and side payments $\overline{u}_i$ identically set to zero. Furthermore, these prices can be computed in $\mathcal{O}(m^3n)$ time.*

Note that, given an **IC** allocation rule, **X** the nominal expected revenue,

$$\mathbb{E}_{(\mathbf{v},\mathbf{k})}\Big[\sum_{i=1}^{n}\sum_{j=1}^{k_i} c_{ij}\nu_i(v_i,k_i)X_{ij}(\mathbf{v}) - u_i^0(\mathbf{v},\mathbf{k})\Big] \tag{3.26}$$

is an upper bound on the optimal expected revenue achievable by **X**. Since pointwise maximum $\mathbf{X}^*$ defined in (3.23) is an upper bound on the optimal expected revenues, setting $\mathbf{X} = \mathbf{X}^*$ in (3.26) gives an upper bound on the achievable optimal expected revenue. We compare the relative performance of the two sub-optimal mechanisms proposed in the numerical study presented in the next section, which is remarkable for the synthetic data set used.

Finally, we remark that the problem of finding optimal mechanism is hard because we need to select both an allocation rule and a set of side payments independently. This is because the transfer payment implementing a given **IC** allocation rule is not uniquely defined by the rule. We conclude this section with the following result.

Lemma 3.9. *Suppose the click-through-rates are separable. Then the CRB mechanism is optimal for social surplus maximization. When the virtual surplus is non-decreasing in both v_i and k_i, the CRB with the virtual valuations, $\nu_i(v_i)$ is optimal for revenue maximization.*

3.5 Comparing the Mechanisms: A Numerical Study

In this section, we report the results of a numerical study that compares the revenue and efficiency properties of the **RB** and **CRB** mechanisms as compared to the optimal mechanisms in the slot independent valuation model and the two proposed suboptimal mechanisms as compared to the optimal mechanism in the slotted model. We consider an adword auction model with 4 slots and 6 bidders, i.e.

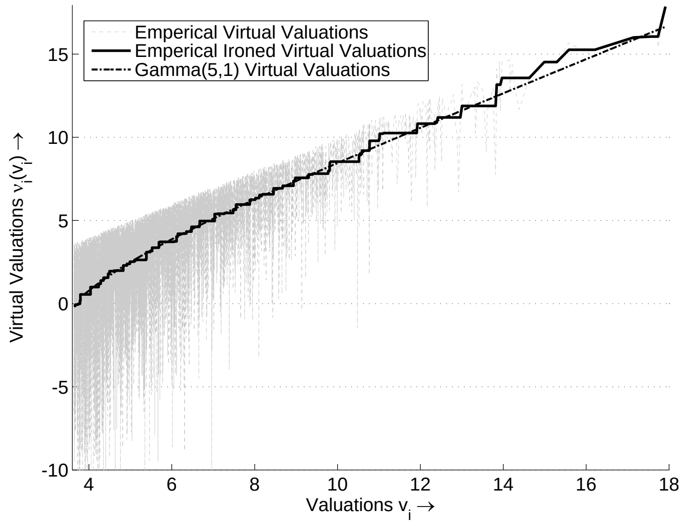


Figure 3.4: Ironed out empirical virtual valuations vs. GAMMA(5,1) virtual valuations

$m = 4, n = 6$. The click-through-rate matrix is given by

$$\mathbf{C} = \begin{bmatrix} 96 & 93 & 47 & 42 \\ 90 & 75 & 24 & 3 \\ 83 & 62 & 19 & 7 \\ 50 & 45 & 42 & 36 \\ 95 & 90 & 82 & 63 \\ 93 & 80 & 77 & 2 \end{bmatrix}$$

where c_{ij} denotes the expected number of clicks per day when bidder i is assigned to slot j. We take the common random number approach[7] and compare the average performance on a sample of $N = 10,000$ valuation vectors. For each bidder $i = 1, \ldots, 6$, the synthetic valuations are generated from a gamma distribution with mean 5 and standard deviation $\sqrt{5}$, i.e. $v_i \sim \text{GAMMA}(5, 1)$.

[7]Common random number approach is a variance reduction simulation technique where alternative system designs are compared based on their average performance on the same set of sampled data.

For virtual surplus maximization, we used the virtual valuations ν_i^{GAMMA} defined by the GAMMA$(5,1)$ distribution[8] as well as the virtual valuation $\tilde{\nu}_i$ implied by the samples. Let $v_i^{[t]}$ denote the value in the t-th position when the samples $\{v_i^{(t)}\}_{t=1}^N$ are sorted in increasing order. Then

$$\tilde{\nu}_i(v_i^{[t]}) = v_i^{[t]} - (v_i^{[t+1]} - v_i^{[t]})\frac{1-\frac{t}{N}}{\frac{1}{N}} \quad t = 1,\ldots,N-1, \tilde{\nu}_i(v_i^{[N]}) = v_i^{[N]}$$

Since the empirical virtual valuations is not monotone, we iron it using the ironing procedure in Myerson [53] to get monotone ironed empirical virtual valuations $\hat{\nu}_i$. From Figure 3.4 that displays $\tilde{\nu}_i, \hat{\nu}_i$ and ν_i^{GAMMA} in the range of valuation where they are positive it is clear that all these three quantities are very close.

3.5.1 Results for the Uniform Valuation Model

As observed in Example 3.2 computing the optimal expected revenue (efficiency) maximizing rank vectors is not a convex optimization problem and, therefore, cannot be solved efficiently. We circumvent this issue by taking the common random number approach and maximizing the average revenue (or efficiency) over the sampled data. For example, revenue maximizing rank vector

$$\mathbf{w}^* \in \underset{\mathbf{w}\in\mathbb{R}_+^n, w_1=1}{\operatorname{argmax}} \frac{1}{N}\sum_{t=1}^N \left\{ \sum_{i=1}^m \sum_{j=i}^m (c_{[j]^{(t)},j} - c_{[j]^{(t)},j+1}) \frac{\gamma_{[j+1]^{(t)}}^{(t)}}{w_{[j]}^{(t)}} \right\},$$

where $\gamma^{(t)} = (w_1^* v_1^{(t)}, \ldots, w_n^* v_n^{(t)})$, $\mathbf{v}^{(t)}$ denotes the valuation vector for the t-th sample, and $[j]^{(t)}$ denotes the index of bidder ranked at the j-th position in the t-th sample. For a given a ranking vector and sample $\mathbf{v}^{(t)}$, the slot prices can be computed in $\mathcal{O}(n\log(n) + m^2)$ time; therefore, one can attempt optimizing the rank vector $\mathbf{w}$ using a derivative-free NLP method. We use a MATLAB based non-linear optimizer to solve these problems. Since the optimization problem is unconstrained

[8] ν_i^{GAMMA} is increasing since Gamma(5,1) is strictly log-concave.

and low dimensional, the MATLAB code is quite stable and converges quickly. The efficiency maximizing rank vector was also computed from the samples.

Table 3.1 displays the average revenue and efficiency for the following mechanisms.

1. The revenue maximizing **RB** mechanism that uses the sample-based optimal rank vector $\mathbf{w}^*$.

2. The efficiency maximizing **RB** that uses the sample-based optimal rank vector $\mathbf{w}^e$.

3. heuristic **RB** mechanism: this mechanism uses the Google rank vector $w_i = \frac{c_{i,1}}{c_{1,1}}$ for efficiency maximization and Yahoo rank vector (i.e. $\mathbf{w} = \mathbf{1}$) for revenue maximization.

4. The **CRB** mechanism: for efficiency maximization the **CRB** ranks using the valuations v_i and for revenue maximization the **CRB** ranks using the virtual valuations $\nu^{\text{GAMMA}}(v_i)$ and $\hat{\hat{v}}_i(v_i)$.

5. The optimal mechanism for the true prior $\text{GAMMA}(5, 1)$ and the empirical prior distribution.

The superscripts "$*$" and "e" denote the revenue maximizing mechanisms and efficiency maximizing mechanism respectively. In the cells corresponding to the **CRB** and the optimal mechanism in Table 3.1, the number inside (resp. outside) the bracket denotes the value obtained by using the virtual valuation $\hat{v}_i$ (resp. ν_i^{GAMMA}). For the optimal **RB** mechanisms the revenue and efficiency was computed using the same set of N samples that were used to compute the rank vectors.

The optimal **RB** achieves 91.95% (92.24%) of the optimal revenue – in contrast, the **CRB** rule achieves 95.75% (95.71%), thus providing a 3.80% (3.47%) improvement in revenues. Similarly, optimal **RB** achieves 87.53% of the optimal efficiency and the **CRB** achieves 93.15% of the optimal efficiency; thus, improving efficiency by 5.62% over the optimal **RB**.

	Revenue Maximization		Efficiency Maximization	
	Π^*	S^*	Π^e	S^e
Heuristic **RB**	998.24	1523.91	938.70	1463.47
Optimal **RB**	1020.30	1558.90	1002.90	1571.40
CRB	1062.45(1058.62)	1585.85(1595.81)	829.36	1672.30
Optimal	1109.58(1106.08)	1687.53(1697.96)	1000.93	1795.24

Table 3.1: The revenue and efficiency in **RB**, **CRB** and optimal mechanisms for uniform valuation model.

Table 3.2 and 3.3 displays the optimal rank vectors, average slot prices and the advertiser's surplus under both efficiency maximizing and revenue maximizing mechanisms for optimal **RB** and heuristic **RB** respectively. Table 3.4 and 3.5 presents the same statistics for the **CRB** and the optimal mechanism respectively. These numerical results support the following observations.

a) The efficiency maximizing rank vector is more *biased* (i.e. away from **1**) as compared to the revenue maximizing rank vector. This is consistent with what was observed earlier in Example 3.2.

b) The **CRB** mechanism tailored for revenue maximization results in higher average prices for each slot and is, therefore, able to extract higher surplus.

c) The **CRB** mechanism tailored for efficiency maximization results in low slot prices and, consequently, low revenues. Thus, it is important that we maximize the virtual surplus when **CRB** is used for revenue maximization.

d) The revenue generated by the **RB** mechanisms when maximizing surplus and virtual surplus is comparable indicating that **RB** is a robust mechanism.

Optimal **RB**						
	R.M.			E.M.		
	w^*	u^*	p^*	w^e	u^e	p^e
1	1	105.19	4.5549	1	110.17	4.3125
2	1.1413	92.29	4.3256	1.2949	123.12	4.1794
3	1.1420	80.74	3.9090	1.0524	66.43	3.7919
4	0.8196	44.65	3.5485	0.6457	24.23	3.4902
5	0.9395	115.47		0.9538	126.82	
6	1.0462	100.24		1.1093	117.67	

Table 3.2: The average slot prices and advertiser surplus in optimal **RB** mechanism for the uniform valuation model.

Heuristic **RB**				
	R.M.		E.M.	
	u^*	p^*	u^e	p^e
1	140.09	4.3294	106.55	4.5954
2	91.73	4.1312	67.22	4.3611
3	68.28	3.7273	59.22	3.9414
4	71.54	3.4333	69.64	3.5342
5	32.81		131.50	
6	120.34		91.54	

Table 3.3: The average slot prices and advertiser surplus in the heuristic **RB** mechanism for the uniform valuation model.

3.5.2 Results for the Slotted Model

We assumed that the privately known slot value $k_i \sim \text{unif}\{1, \ldots, 4\}$ and generated $N = 10000$ samples. We used the same set of samples for the valuation as in the previous subsection.

Table 3.6 displays the average revenue and efficiency with the pointwise maximizer allocation rule and **CRB** allocation rule. Table 3.7 and Table 3.8 displays the average advertiser surplus, average slot prices and average side payments to each bidder in the pointwise maximizer and **CRB** mechanisms respectively. Note that the total average side payment is remarkably low 6.40 (3.3237) when com-

CRB				
	R.M.		E.M.	
	u^*	p^*	u^e	p^e
1	121.93(125.53)	4.7052(4.6609)	185.92	3.8542
2	61.26(62.58)	4.3955(4.3441)	56.19	3.3017
3	48.23(46.41)	3.8203(3.7871)	30.09	2.2039
4	58.45(62.89)	3.2472(3.2937)	115.06	1.6034
5	130.23(133.05)		280.58	
6	103.31(106.72)		175.12	

Table 3.4: The average slot prices and advertiser surplus in **CRB** mechanisms for the uniform valuation model.

Optimal				
	R.M.		E.M.	
	u^*	p^*	u^e	p^e
1	115.89(119.26)	4.4409(4.4139)	181.92	4.1223
2	97.09(99.69)	4.1567(4.1124)	114.42	3.2994
3	86.80(84.74)	3.7102(3.6775)	81.86	2.8008
4	51.31(55.22)	2.9123(2.9355)	52.82	2.9273
5	117.65(119.83)		194.21	
6	109.22(113.14)		169.08	

Table 3.5: The average slot prices and advertiser surplus in the optimal mechanisms for the uniform valuation model.

pared to the average revenue. The data in the table together with the bound given by equation (3.26) implies that 988.61+6.40=995.01 (984.9536+3.3237= 988.2763) is an upper bound on the achievable revenue. The pointwise maximizer achieves 99.36% (99.66%) revenue of this bound, while the **CRB** achieves 94.78% (95.05%) revenue of this upper bound. Thus, both proposed mechanisms are likely to be close to optimal. For efficiency, **CRB** achieves 92.79% of the true optimal VCG (or pointwise maximizer with $\nu_i(v_i) = v_i$) efficiency as shown in Table 3.6.

Table 3.7 and Table 3.8 presents the average side payments, average advertiser surplus and the average slot prices per click for both the mechanisms. The prices tabulated are not adjusted for side payments. This is because side payments could be positive even when no slot is allocated to the advertiser (this happens when at a particular bid profile the advertiser can get a positive allocation by underbidding k_i; however, no slot is allocated at the true k_i). We found that the distribution of side payments is very skewed, i.e. even though the average side payments are very small, at certain bid profiles the side payments are of the same order of magnitude as the advertiser surplus.

	R.M.		E.M.	
	Π^*	S^*	Π^e	S^e
Pointwise Maximizer	988.61 (984.95)	1468.91(1475.55)	806.50	1562.49
CRB	943.11 (939.37)	1385.20 (1392.49)	695.64	1449.87

Table 3.6: The revenue and efficiency in **CRB** and optimal mechanisms for the slotted model.

Pointwise Maximizer					
	R.M.			E.M.	
	u^*	p^*	$\overline{u}^*$	u^e	p^e
1	97.68(100.15)	4.9157(4.8965)	1.7401(0.9592)	164.73	4.6090
2	78.55(80.70)	4.3183(4.2910)	0.8961(0.6066)	101.03	3.4710
3	66.60(65.06)	3.1226(3.1033)	0.6893(0.5558)	69.42	1.6009
4	37.36(39.61)	1.0297(1.0457)	0.4743(0.1776)	58.29	0.2571
5	101.78(103.31)		1.6600(0.5830)	195.57	
6	91.93(95.12)		0.9417(0.4405)	166.95	

Table 3.7: The average slot prices and advertiser surplus in the **CRB** mechanisms for the slotted model.

3.6 Conclusion

The discrete structure of the adword auction allocation space allows us to reduce the **IC** constraint to the existence of bidder dependent slot prices at which each bid-

CRB				
	R.M.		E.M.	
	u^*	p^*	u^e	p^e
1	114.61(118.04)	5.0799(5.0405)	196.07	4.2767
2	61.77(63.17)	4.243(4.2082)	73.19	3.0186
3	48.47(46.51)	2.7492(2.7484)	46.36	1.1265
4	35.84(38.51)	1.0054(1.0260)	71.52	0.0888
5	97.02(99.04)		207.09	
6	84.38(87.86)		160.00	

Table 3.8: The average slot prices and advertiser surplus in the optimal mechanisms for the slotted model.

der self-selects their allocated slot. We use this new characterization to show that in this auction models there are **IC** mechanisms which are not affine maximizers. Achieving optimal revenues with multi-dimensional types in this model remains a hard stochastic program involving multi-dimensional ironing and sweeping (Rochet and Chone [59]).

In slot independent private valuation model, the pricing characterization collapses to the existence of ordered set of thresholds at which a given advertiser is allocated particular slots. The prices implementing any **IC** allocation rule can by computed very efficiently. When the click through rate is not separable, we show that the revenue (efficiency) maximizing rank vector can be efficiently computed using the history of bids available for each adword. When prior information is unavailable, the proposed **CRB** allocation rule is a very attractive choice since it is simple, non-parametric, computationally efficient and has a superior performance to any rank based mechanism. Our numerical study also indicate that both **RB** and **CRB** achieve close to optimal revenues and efficiency without using detailed prior information.

In the slotted auction model even though the set of **IC** mechanism is easy to characterize, computing the optimal mechanism problem is hard because of the non-zero side payments needed to screen the slot information. Our numerical study indicates that the two sub-optimal mechanisms proposed are likely to perform close to optimal.

Chapter 4

An Equilibrium Model for Matching Impatient Demand and Patient Supply over Time

4.1 Introduction

Standard strategic models for dynamic economies have had limited success in predicting the real world market dynamics mainly because of the following two reasons: first, it is difficult to analyze a dynamic market with complex dynamics, consequently models that are solvable are necessarily simple abstractions that unable to incorporate many important features of "real" markets; second, in order to remain tractable, economic models need to assume that agents are rational; however, rationality imposes such unreasonable demands that agents are almost never rational in "real" markets.

To overcome these issues, many models (see, e.g. Daniels et al. [17]; Farmer et al. [23]; Farmer and Zovko [24]; Luckock [47]) for dynamic markets assume that the actions of the agents are randomly distributed according to a distribution that is chosen to reflect nominal economic behavior of agents. In such models, the interesting features of the statistical behavior of the market is a consequence of the market dynamics itself.

In this chapter, we present a model that incorporates features of both of the strategic and randomized approaches. In our model, the agents are strategic, i.e. they do not take random actions; however, agents have bounded rationality, and therefore, do not base their actions on the detailed market conditions at the arrival epoch but on the average long term market characteristics arising from the random actions of other agents. In effect, we define an equilibrium concept on the *distribution* induced by the random agent actions. In this equilibrium in the long run the essential market information filters to the agents and gets reflected in the statistical properties of the market.

There is a large body of literature investigating financial markets with the limit order book as a market clearing mechanism. Farmer and Zovko [24] demonstrate a striking regularity in the way people place limit orders in financial markets using a data set from the London Stock Exchange. They define the relative limit price as the difference between the limit price and the best price available for instantaneous execution. They conclude that for both buy and sell orders, the unconditional cumulative distribution of relative limit prices decays roughly as a power law with an exponent approximately $\beta \approx 1.5$, i.e. $\mathbb{P}(\text{limit price} > x) = \frac{A}{(x_0+x)^\beta}$. This behavior spans more than two decades, ranging from a few ticks to about 2000 ticks. Bouchaud et al. [11] also report a power law distribution of prices. Our model shows that this type of power tail distribution can arise in equilibrium for a limit order book mechanism when more patient trader get better price in order for less patient trader to get better execution time.

The models in Luckock [47]; Mendelson [49], and Domowitz and Wang [20] are not strategic in that they assume a stationary order arrival independent of the state of the book. This can be reasonable in very fast highly liquid market, where for small time horizon, large number of traders participate in the trading process, and, their strategies average out, resulting in a random stationary behavior. In addition, these models assume that the order arrival pattern is *exogenous* to the model; consequently, their approach reduces to performance evaluation for a given supply and demand function. This approach leads to many pathologies, such as accumulation of orders outside the active window (see Luckock [47]). In contrast, in our model the stationary equilibrium behavior of traders gives rise to *endogenous* limit order distribution. Our model is similar to the separable markets discussed in Luckock [47].

4.2 Continuous Online Exchange

We consider a continuous time online exchange market for a single commodity or a set of substitutable commodities. There is no fee for using the online exchange and orders cannot be canceled. This exchange only allows the posted prices to be multiples of a tick size ϵ. We assume that there exists a high enough constant price $(N+1)\epsilon$ at which an exogenous seller is willing to sell unlimited amount of the commodity and, symmetrically, a low enough price (normalized to 0) at which an exogenous buyer is willing to buy an unlimited quantity. Thus, the set of allowed selling prices is a discrete set $\{\epsilon, 2\epsilon, \dots, N\epsilon\}$.

The sellers on this exchange arrive according to a Poisson process with rate λ. Each seller offers only one unit of the commodity for sale. The sellers are assumed to be risk neutral and incur a cost that is proportional to their execution time, i.e. the time elapsed between their arrival and sale of their item. On arrival, the sellers choose a selling price, or equivalently a price tick j, based on their beliefs about expected time to trade execution at each price tick. We assume that sellers believe that the expected time to execution depends *only* on the selling price; thus, excluding the possibility of sellers exploiting the complete state of the online exchange, specifically the number of unsold items at each tick. Note that we are implicitly assuming that sellers only react to the *long-term average impact* of the actions of all the agents using the exchange.

The buyers on this exchange arrive according to an independent Poisson process with rate μ. On arrival, each buyer independently decides to buys one unit with probability $\beta(p)$ when the current lowest (outstanding) selling price is p, where the demand function $\beta(p)$ is downward slopping, i.e. the $\beta(p)$ function is non-increasing in p. Buyers who arrive to find no sellers are lost. Note that our model assumes that the sellers are *patient* whereas the buyers are *impatient*.

The expected profit u_n of the n-th arriving seller is given by

$$u_n(j_n, \delta_n) = j_n\epsilon - \delta_n T_n(j_n) \tag{4.1}$$

where j_n denotes the price-tick selected by the seller, δ_n is the *patience* parameter of the seller, and $T_n(j_n)$ is the expected execution time at price tick j_n. We have normalized the cost (or exogenous value) of the commodity to the seller to zero. The patience parameters $\{\delta_n : n \geq 1\}$ are assumed to be independent and identically distributed (IID) samples from a distribution function F_δ, which is assumed to be continuous with support $[0, \bar{\delta}]$. Thus, the sellers are heterogeneous in their patience parameter. We assume that this heterogeneity in patience is a manifestation of the heterogeneity in the agents' own business models. The utility function in (4.1) is motivated by the money value of time (e.g. lost labor, cost of tracking the trade etc) rather than time value of money (i.e. delayed payments). See Foucault et al. [28] for a detailed discussion on this utility function. We expect that the results in this chapter to hold for a large class of utility function that are monotone in price and execution time.

In our model, we ask what equilibrium supply function would result from the interaction of sellers heterogenous in their patience and impatient buyers in a stationary online exchange market environment. In particular, we are interested in how this supply function depends on the market parameters such as demand elasticity and traffic intensity.

4.2.1 Information Structure

We assume that market mechanism is common knowledge i.e. all sellers know that a sequence of sellers would arrive according to a Poisson process and offer their unit of supply at price that optimizes their own utility function given by (4.1). The patience parameter distribution F_δ is common knowledge among the sellers, whereas the patience parameter δ_n is a private information of seller n.

Sellers form beliefs about the expected execution time $T_n(j)$ at the price tick j based on the their information set and post their unit at the price

$$\sigma_n(\delta) = \underset{j \in \{1,2\ldots,N\}}{\operatorname{argmax}} \left\{ j\epsilon - \delta T_n^{\sigma_{-n}}(j) \right\}, \tag{4.2}$$

where the notation $T_n^{\sigma_{-n}}(j)$ indicates that the belief about the execution time depends on the action of all other sellers. Note that we assume that the sellers beliefs about the expected execution time depends *only* on the price tick j.

Since the information sets are symmetric, we restrict ourselves to symmetric equilibrium strategies. Thus, the equilibrium beliefs of all sellers is symmetric, i.e. $T_n^{\sigma_{-n}}(j) = T(j)$. From this point onward, by equilibrium and beliefs we mean symmetric equilibrium and symmetric beliefs.

Definition 4.1. *A selling strategy $\sigma^*(.)$, that maps a patience parameter δ to a selling price tick in $\{1, 2, \ldots, N\}$, is a Bayesian Nash Equilibrium (BNE) if it solves (4.2) when $T^{\sigma^*}(j)$ is the stationary expected execution time given that all future trader follow the strategy σ^*.*

The execution time $T_n^{\sigma_{-n}}$ in (4.2) has a two-fold expectation: first, given the sample path of patience parameter δ_n of arriving sellers, it is expectation over the arrival process of the buyers; and second, there is the expectation over all sample paths of patience parameters. The latter expectation give rise to the notion of Bayesian Nash equilibrium. Since the patience parameters of sellers are assumed to be independent, it follows that the knowledge of the draw δ_n does not give seller n any information about the patience parameter of the other sellers.

4.2.2 Simple Example

The following example illustrates our proposed model and the nature of results we establish later in this section.

Two sellers with one unit each of a given product arrive at time $t = 0$ into a market place where sellers are restricted to sell their product at a price $p \in \{p_1, p_2\}$, $(p_1 < p_2)$. Buyers arrive according to a Poisson process with μ_1 (resp. μ_2) if the cheapest unit of the product is available at price p_1 (resp. p_2).

The sellers decide their selling prices as a function of their waiting cost rate δ at time 0 and then the market clears. Suppose at a symmetric equilibrium[1], a seller posts the price p_j, $j = 1, 2$, whenever the cost coefficient $\delta \in \mathcal{B}_j$, where $\mathcal{B}_1 \cup \mathcal{B}_2 = [0, \bar{\delta}]$. Let $\alpha_j = F_\delta(\{\mathcal{B}_1\})$ denote the equilibrium probability of selecting price p_1 at this equilibrium.

Suppose whenever there are two sellers at a given price, each seller receives the next order with probability $\frac{1}{2}$. Then the expected waiting time if a seller selects price p_1 (resp. p_2) is $\frac{\alpha_2}{\mu_1} + \frac{3\alpha_1}{2\mu_2}$ (resp. $\frac{3\alpha_2}{2\mu_2} + \alpha_1(\frac{1}{\mu_1} + \frac{1}{\mu_2})$).

[1]Symmetric Bayesian Nash equilibrium of the one shot game with δ_n as private information to be precise.

Since we assume that sellers only react to *long-term average* values, a seller would choose price p_1 if, and only if,

$$p_1 - \delta\big[\alpha_2 \frac{1}{\mu_1} + \alpha_1 \frac{3}{2\mu_2}\big] > p_2 - \delta\big[\alpha_2 \frac{3}{2\mu_2} + \alpha_1 (\frac{1}{\mu_1} + \frac{1}{\mu_2})\big],$$

i.e.

$$\delta > \frac{\mu_1 \mu_2 (p_2 - p_1)}{\mu_1 - \mu_2 + \frac{\alpha_2}{2}(\mu_1 + \mu_2)}.$$

Since, at equilibrium, the probability of choosing price p_2 is α_2, it follows that

$$\alpha_2 = F_\delta \left(\frac{\mu_1 \mu_2 (p_2 - p_1)}{\mu_1 - \mu_2 + \frac{\alpha_2}{2}(\mu_1 + \mu_2)} \right) \tag{4.3}$$

The solution to (4.3) uniquely characterizes the equilibrium.

Suppose $\delta \sim \text{unif}[0,1]$, i.e. $F_\delta(x) = x$, for all $x \in [0,1]$. Then at equilibrium

$$\alpha_2 = \frac{\sqrt{(\mu_1 - \mu_2)^2 + 4(\mu_1 + \mu_2)\mu_1 \mu_2 (p_2 - p_1)} - 2(\mu_1 - \mu_2)}{\mu_1 + \mu_2}$$

Note that the three factor affecting this equilibrium are: the price tick size $\epsilon = p_2 - p_1$, the demand elasticity $\mu_1 - \mu_2$ and the distribution F_δ.

4.2.3 Stationary Equilibrium Outcomes

In order to characterize the BNE we need to be able to solve for the stationary behavior of the exchange for a given strategy σ. We show below that the exchange is a Markovian priority queuing system for which the waiting time can be computed in closed form.

Since $\{\delta_n : n \geq 1\}$ are IID, (4.2) implies that a selling strategy σ *thins* the arriving sellers according to some probability mass function α_j^σ, $j = 1, \ldots, N$, i.e. the arrival rate of sellers that post their unit at the price-tick j is $\lambda \alpha_j^\sigma$, $j = 1, \ldots, N$.

Let $X_n(j)$ denote the inventory of outstanding orders at price tick j when the n-th seller arrives in the exchange, and let $\mathbf{X}_n = (X_n(1), ..., X_n(N))$. The dynamics described in Section 4.2 implies that $\mathbf{X}_n$ is the queue length process for the Markovian preemptive priority queuing system with N customer classes where the customer class j has an arrival rates $\lambda\alpha_j^\sigma$ and a service rate $\mu\beta_j$ where $\beta_j \triangleq \beta(j\epsilon)$ is the probability of purchase by an arriving buyer if cheapest unit is available at price tick j, and a class j customer has priority over a class i customer whenever $i > j$. It follows that the properties of the state process $\{\mathbf{X}_n : n \geq 1\}$ are completely determined by thinning probabilities $\{\alpha_j^\sigma\}$. In order to emphasize this fact, we drop the superscript σ and index variables by α.

For $j = 1, \ldots, N$, define the traffic intensity in customer class j as

$$\rho_j^\alpha \triangleq \frac{\lambda}{\mu} \cdot \frac{\alpha_j}{\beta_j}$$

The following lemma gives a closed-form solution for the steady state expected execution time T_j^α at the price tick j.

Lemma 4.1. *For $j = 1 \ldots N$, if $\sum_{i=1}^{j} \rho_i^\alpha < 1$, then the expected execution time at price tick j is finite and is given by*

$$T^\alpha(j) = \frac{1}{\mu}\left(\frac{1}{1 - \sum_{i=1}^{j-1} \rho_i^\alpha}\right)\left[\frac{1}{\beta_j} + \frac{\sum_{i=1}^{j}\left(\frac{\rho_i^\alpha}{\beta_i}\right)}{1 - \sum_{i=1}^{j} \rho_i^\alpha}\right] \tag{4.4}$$

otherwise the queue at the price tick j grows beyond bound and $T^\alpha(j) = \infty$.

The proof of this Lemma can be found in Resing and Adan [57] (pp 90, equation 9.5). Little's law implies that the long-term average expected inventory Q_j^α of unsold items at price tick j is given by

$$Q_j^\alpha \triangleq \mathbb{E}\, X_\infty^\alpha(j) = \lambda\alpha_j T^\alpha(j) \tag{4.5}$$

where $\mathbf{X}_\infty^\alpha = \lim_{n\to\infty} \mathbf{X}_n^\alpha$ if it exists. Since $\rho_j^\alpha \geq 0$, and β_j is decreasing in j, the expected execution time $T^\sigma(j)$ is non-decreasing in j for all α. Thus, the sellers face the following tradeoffs - a better execution price can only be obtained at the cost of a larger expected waiting time at the exchange.

Next, we characterize the average inventory seen by the buyers or sellers arriving to the exchange when the current outstanding price-tick is j, or equivalently, $\sum_{l=1}^{j-1} X_\infty^\alpha(l) = 0$.

Lemma 4.2. *Let $X_\infty(j)$ denote the steady state number of class j customers in Markovian n-class priority queue with preemption. Then*

$$\mathbb{E}\Big[\sum_{l=j}^{k} X_\infty(l)\Big|\sum_{l=1}^{j-1} X_\infty(l) = 0\Big] = \frac{\rho_{jk}(1-2\rho_{1,j-1}+\rho_{1,j-1}^2+\rho_{1,j-1}\rho_{jl})}{(1-\rho_{1,j-1})^2(1-\rho_{1,j-1}-\rho_{jl})}, \tag{4.6}$$

where $\rho_{ij} = \sum_{k=i}^{j} \frac{\lambda\alpha_k}{\mu\beta_k}$, λ_k is the arrival rate of class k customers and μ_k is the service rate of the class k customers.

Proof: Since the distribution of $\sum_{l=j}^{k} X_\infty(l)$ and $\sum_{l=1}^{j-1} X_\infty(l)$ does not depend on the service discipline *within* the sets $\{1,\ldots,j-1\}$ and $\{j,\ldots,k\}$, it suffices to prove (4.6) for a queue with two customer classes and traffic intensities $\rho_2 = \rho_{jl}$ and $\rho_1 = \rho_{1j}$.

The generating function for the joint distribution of $(X_\infty(1), X_\infty(2))$ is given by (see (3.15) on pp. 95 in Jaiswal [38])

$$\begin{aligned}\Pi(z_1,z_2) &= (1-\rho_1-\rho_2)\Bigg[1+\Big(\frac{\lambda_1\{z_1-\bar{b}_1[\lambda_2(1-\alpha_2)]\}}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}\Big)\times \\ &\quad \Big(\frac{1-\bar{S}_1\{\lambda_1(1-z_1)+\lambda_2(1-z_2)\}}{1-\frac{1}{z_1}\bar{S}_1\{\lambda_1(1-z_1)+\lambda_2(1-z_2)\}}\Big)\Bigg]\times\frac{(z_2-1)\bar{c}_2\{\lambda_2(1-z_2)\}}{z_2-\bar{c}_2\{\lambda_2(1-z_2)\}}\end{aligned}$$

where

$$\begin{aligned}\bar{b}_1(s) &= \frac{\mu_1+\lambda_1+s-\sqrt{(\mu_1+\lambda_1+s)^2-4\lambda_1\mu_1}}{2\lambda_1}, \\ \bar{S}_i(s) &= \frac{\mu_i}{\mu_i+s} \quad i=1,2, \\ \bar{c}_2(s) &= \bar{S}_2\{\lambda_1(1-\bar{b}_1(s))+s\}.\end{aligned}$$

We obtain the generating function $\Pi(z)$ for distribution of $X_\infty(2) \mid X_\infty(1) = 0$ by substituting $z_1 = 0$ and $z_2 = z$ in the expression for $\hat{\Pi}(z_1,z_2)$, i.e.

$$\Pi(z) = (1-\rho_1-\rho)\frac{(z-1)\bar{c}_2\{\lambda_2(1-z)\}}{z-\bar{c}_2\{\lambda_2(1-z)\}}$$

The expectation $\mathbb{E}[X_\infty(2)|X_\infty(1)=0] = \frac{\partial \Pi(z)}{\partial z}\Big|_{z=1}$. Simplifying we get,

$$\mathbb{E}[X_\infty(2)|X_\infty(1)=0] = \frac{\rho_2(1-2\rho_1+\rho_1^2+\rho_1\rho_2)}{(1-\rho_1)^2(1-\rho_1-\rho_2)}$$

■

Lemma 4.2 describes the steady state expected inventory the buyers or sellers see when the current price is $j\epsilon$. Although we restrict the sellers to base their beliefs of the execution time on *only* on the price tick j, Lemma 4.2 allows us to reconstruct the *entire* state of the exchange when a seller places an order.

Before characterizing the equilibrium selling strategy, we would like to point out that the distribution of the steady state price s^α that an arriving buyer observes is

$$\mathbb{P}(s^\alpha > j\epsilon) = 1 - \sum_{k=1}^{j} \rho_k \tag{4.7}$$

Note that this distribution is not the same as the distribution of trade execution prices α_j. The distribution s^α is an average over time, and latter, i.e. $\{\alpha_j\}$, is an average over orders. Our model will focus on the latter. Also, it is easy to notice that if the demand is inelastic, i.e. $\beta(p) \equiv 1$, the distribution of s conditional on it being finite is same as the distribution α. We consider inelastic demand in Section 4.3.

By Definition 4.1 a selling strategy σ^* is a symmetric BNE strategy iff

$$\sigma^*(\delta) \in \underset{k\in\{1,\ldots,N\}}{\operatorname{argmax}} \left\{k\epsilon - \delta T^{\sigma^*}(k)\right\}, \tag{4.8}$$

equivalently, a distribution α^* is a symmetric BNE strategy iff

$$\alpha_j^* = \mathbb{P}\left\{ j \in \underset{1\le k\le N}{\operatorname{argmax}} \left\{k\epsilon - \delta T^{\alpha^*}(k)\right\}\right\} \tag{4.9}$$

The following theorem establish the existence of a symmetric equilibrium in this model.

Theorem 4.1. *An equilibrium distribution α^* satisfying (4.9) always exists. Furthermore, suppose the expected execution time $T^{\alpha^*}(j)$ is strictly increasing and convex in j, i.e. for $j = 2, \ldots, N$ the difference $T^{\alpha^*}(j) - T^{\alpha^*}(j-1)$ is non-negative and non-decreasing in j. Then every equilibrium selling strategy σ^* that results in α^* is non-increasing in δ i.e. at the equilibrium α^* the seller with higher waiting cost rate post a lower selling price.*

Proof: Define $\Psi : \mathcal{S}^N \mapsto \mathcal{S}^N$ as follows

$$\Psi_j(\alpha) = \mathbb{P}\left[\delta \in \left\{x \in [0, \bar{\delta}] \middle| j\epsilon - xT^{\alpha}(j) \geq k\epsilon - xT^{\alpha}(k)\ \forall k\right\}\right], \tag{4.10}$$

where $\mathcal{S}^N$ denotes the probability simplex with N support points. Observe that for any α, the set

$$\left\{x \in [0, \bar{\delta}] \mid j\epsilon - xT^{\alpha}(j) \geq k\epsilon - xT^{\alpha}(k)\ \forall k\right\}$$

is either empty or an interval and Ψ_j is the probability of δ belonging to this interval under F_δ. Since the time to execution $T^\alpha : \mathcal{S}^N \mapsto \mathbb{R}_+^N \bigcup \{\infty\}^N$ is continuous[2] function of α and F_δ is assumed to be continuous, Ψ is a continuous function of α. Thus, the Brouwer fixed point theorem implies that Ψ has at least one fixed point. Since a fixed point of Ψ is a solution to (4.9), an equilibrium satisfying (4.9) always exists.

If at an equilibrium α^*, T^{α^*} is convex in j then

$$h^*(j, \delta) = j\epsilon - \delta T^{\alpha^*} \tag{4.11}$$

is concave in j. Also, since $T^{\alpha^*}(j)$ is strictly increasing j, $h(j, \delta)$ is strictly decreasing difference in (j, δ). Thus, Theorem 10.6 in Sundaram [65] implies that $\sigma^*(\delta) \in \operatorname{argmax} h(j, \delta)$ is non-increasing in δ. ■

[2] T^α is clearly continuous every α where it is finite and it approaches ∞ as $\sum_{j=1}^{k} \rho_k^\alpha$ approaches 1 for any $1 \leq k \leq N$ and hence is continuous at the boundary as well.

4.2.4 Numerical Example

In the following example, we numerically compute the equilibrium in a market with elastic demand function.

Suppose $N = 50$, $\epsilon = 1$, $\delta \sim U[0, 160]$, $\lambda = 3$, $\mu = 12$ and the demand function be given by

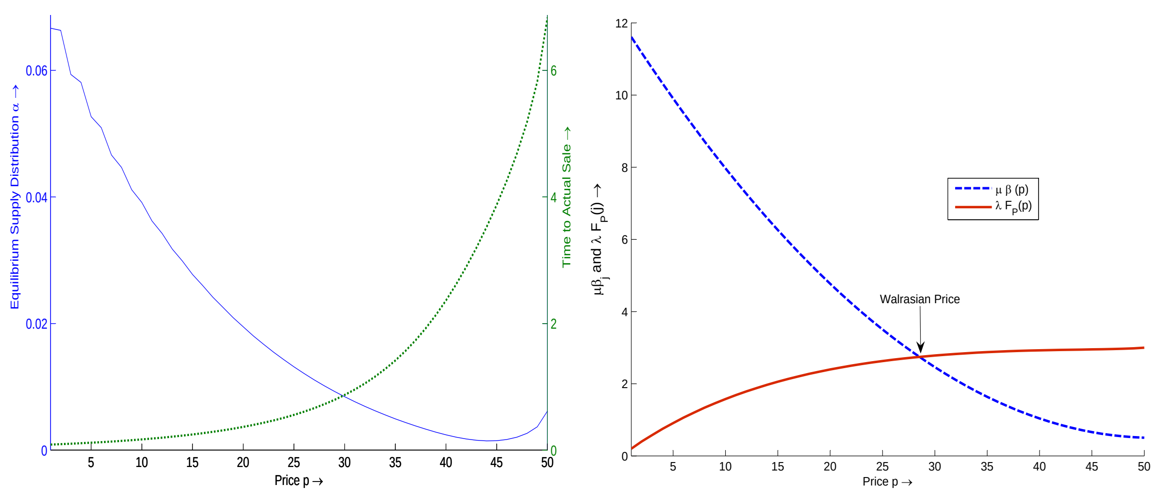
$$\beta_j = \frac{1}{12}\left[0.5 + \left(\frac{N-j+1}{15}\right)^2\right]$$

These parameters are chosen to ensure that the resulting equilibrium has full support.

Let $\Psi : \mathcal{S}^N \mapsto \mathcal{S}^N$ denotes the functions defined in (4.10). Then, Theorem 4.1 implies that the optimization problem,

$$\min_{\alpha \in \mathcal{S}^N} \|\Psi(\alpha) - \alpha\|_2 \tag{4.12}$$

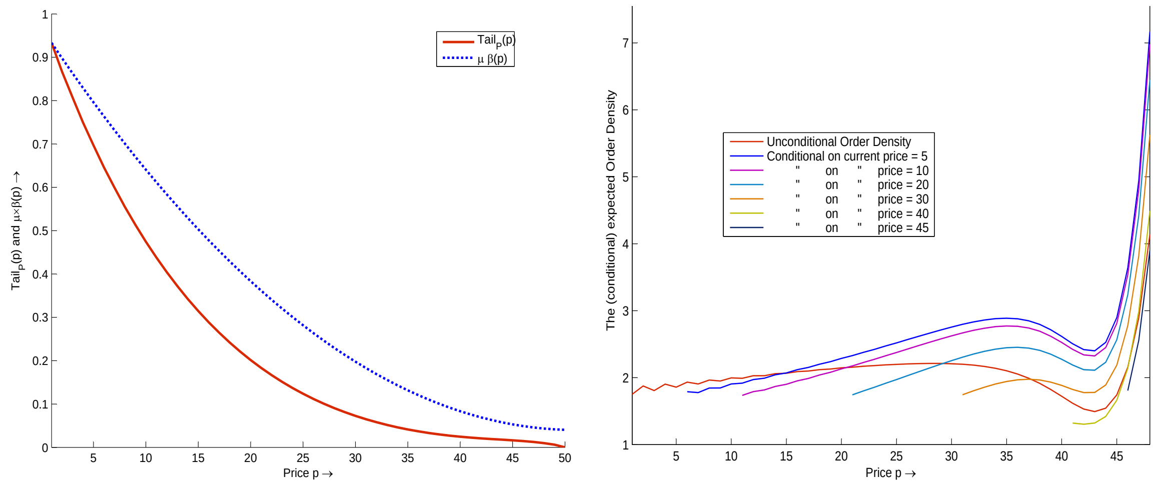
has an optimal value equal to zero, and every optimal solution is an equilibrium. We used MATLAB optimization routine FMINCON to solve optimization problem (4.12).



(a) The equilibrium supply density and the execution time as a function of price

(b) The equilibrium supply function and the demand function

Figure 4.1: Numerical equilibrium: tradeoffs



(a) The demand decay, β_j and equilibrium tail of α as a function of price

(b) The (conditional) expected order densities

Figure 4.2: Numerical equilibrium: order density and supply decay

Figure 4.1(a) displays the expected execution time T^{α^*} and the supply thinning distribution α^*. Figure 4.1(b) displays λ times the equilibrium CDF of α^*, i.e. the supply function, and the demand function $\mu\beta(p)$. For the choice of parameters in this example, the Walrasian market clearing price is approximately 28. Recall that the trading price available to a random buyer, i.e. the time average of the trading price, is distributed according to $\mathbb{P}(s^{\alpha^*} = j\epsilon | s^{\alpha^*} \leq N\epsilon) = \frac{\rho_j}{\sum_k \rho_k}$ corresponding to the tail CDF given in (4.7). For the parameters in this example, this distribution has a mean value of 23.77 and a standard deviation 15.55. Thus, the time-average of the prices is close to the Walrasian market clearing price. On the other hand, the mean and standard distribution of thinning distribution $\{\alpha_j^*\}$, i.e. the average price available to the sellers, are, respectively, 12.70 and 10.52. This is significantly lower than the Walrasian price. A possible explanation for this phenomenon is that, since arriving sellers does not observe current outstanding price s^{α^*}, they post a low price even though the current selling price is higher. These low price orders are traded very fast, and therefore, do not contribute to the time averages.

Figure 4.2(a) plots the supply thinning function $\mathbb{P}(\alpha^* > j)$ and the demand decay β_j as a function of the price-tick j. Notice that the equilibrium supply thinning function $\mathbb{P}(\alpha^* > j)$ decays significantly faster than the demand function β_j. This is because competition from other sellers and the desire to reduce their waiting time induces sellers to post low prices.

Figure 4.2(b) plots the expected inventory of outstanding sellers as a function of the price. The expected inventory is hump shaped as a function of price and increases rapidly at the right boundary because of the boundary effects, which is empirically observed in the context of limit order book in Bouchaud et al. [11]. We also observe that the shape of expected inventory conditional on current selling price is essentially independent of the current price. We expect this not to be the case if the sellers are allowed to condition their selling strategy on the current price because a seller facing a high current price would undercut more frequently than in this model resulting in smoother boundary in the expected order density at the current price.

4.3 Exchange with Inelastic Demand

In previous section, we observed that the demand elasticity has a serious impact on the equilibrium price distribution because sellers with low patience parameter are unable to force the buyers to pay a high price. In this section, we assume that the demand is inelastic in price, i.e. $\beta(p) = 1$ for all $p \in (0, \infty)$. Later in this section we show that an inelastic demand model is reasonable for most commodities in markets with very high trade frequency. Our main goal in this section is to investigate whether competition between sellers is sufficient to maintain reasonable (low) prices in a market with inelastic demand. The results in this section settle this question in the negative, i.e. sellers are able to leverage their market power to set very high prices. This result is consistent the result in Borenstein [10] which finds that inelastic demand in conjunction with an continuous exchange mechanism resulted in unprecedented high prices in the California electricity markets.

In this section, we work with continuous prices. We derive the differential equation characterizing the equilibrium thinning rate $\alpha^*(p)$ by taking appropriate limits in (4.9). Let $\epsilon \to 0$, $N \to \infty$, such that $N\epsilon \to \infty$ and the selling pricing grid

$$\{\epsilon, 2\epsilon, \ldots, N\epsilon\} \to (0, \infty).$$

Let $\alpha_P(p)$ and F_P denote, respectively, the density and the CDF of the equilibrium selling price. Taking the limit in (4.11), the seller's expected surplus is given by

$$h(p,\delta) = p - \frac{\delta}{\mu}\left[\frac{\rho F_P(p)}{(1-\rho F_P(p))^2} + \frac{1}{(1-\rho F_P(p))}\right] \tag{4.13}$$

Since we focus on explicitly constructing an equilibrium, we do not focus on whether an equilibrium exist i.e. whether the continuous version of (4.9) has a solution. After differentiating $h(p,\delta)$ with respect to p and simplifying, the first order conditions are given by

$$\delta(p) = \frac{\mu}{2} \cdot \frac{(1-\rho F_P(p))^3}{\rho \alpha_P(p)} \tag{4.14}$$

Assume that $\sigma^*(\delta)$ is monotone in δ. We later verify that in the equilibrium that we construct this indeed is the case.

The equilibrium conditions are equivalent to the following conditions on the tail of the selling price distribution F_P:

$$\mathbb{P}(\text{price } > p) = \mathbb{P}(\delta \leq \delta(p)) \quad \forall p \tag{4.15}$$

In order to obtain a closed form solution, we assume that the patience parameter δ is distributed uniformly on $[0, \bar{\delta}]$. Thus, (4.14) is equivalent to the following implicit point-wise condition on the F_P:

$$\alpha_P(p) = \frac{d}{dp}F_P(p) = \frac{\mu}{2\overline{\delta}} \cdot \frac{(1-\rho F_P(p))^3}{\rho(1-F_P(p))} \tag{4.16}$$

Solving the above ordinary differential equation (ODE) implies the following result.

Lemma 4.3. *The equilibrium price distribution, i.e. the solution to the ODE (4.16), is given by*

$$F_P^*(p) = \begin{cases} 0, & p \leq 0, \\ \frac{\rho(1+\frac{\mu}{\overline{\delta}}p)-\sqrt{1-(1-\rho)(1+\rho(1+\frac{\mu}{\overline{\delta}}p))}}{\rho(1+\rho(1+\frac{\mu}{\overline{\delta}}p))}, & 0 \leq p \leq K, \\ 1, & p \geq K, \end{cases}$$

where the support K of the equilibrium selling price distribution is given by

$$K = \frac{\overline{\delta}}{\mu} \cdot \frac{\rho}{1-\rho}. \tag{4.17}$$

As $\rho \longrightarrow 1$, the support $K \longrightarrow \infty$ and

$$lim_{\rho \to 1} F_P^*(p) = 1 - \frac{2\overline{\delta}}{2\overline{\delta}+\mu p}, \qquad p \geq 0. \tag{4.18}$$

Using (4.16) and (4.14) we get that the equilibrium selling strategy

$$\sigma^*(\delta) = F_P^{*-1}\left(1 - \frac{\delta}{\overline{\delta}}\right)$$

Since, F_P^* is continuous on $[0, K]$ in Lemma 4.3 this equilibrium satisfy our assumption that σ^* is monotone in δ over $[0, \overline{\delta}]$.

Lemma 4.3 establishes that as the congestion ρ increases, the distribution of prices approach a power-law distribution, i.e. the equilibrium price distribution has very heavy tails even when the patience parameter δ is uniformly distributed between 0 and a finite upper bound $\overline{\delta}$. Thus, in congested markets with inelastic demand the sellers are able to leverage their market power to set (and obtain) very high prices. This partially explains the phenomena observed in California electricity markets Borenstein [10].

The stationary expected inventory $Q(p)$ at price p is given by

$$Q(p) = \frac{\partial}{\partial p}\left\{\frac{\rho F_P(p)}{(1-\rho F_P(p))}\right\} = \frac{\mu}{2\bar{\bar{\delta}}}\left(1 + \frac{(1-\rho)F_P(p)}{1-F_P(p)}\right). \tag{4.19}$$

We use Lemma 4.2 to compute the conditional expected density of outstanding orders. For all $p \geq s$, the conditional expected seller density

$$Q^c(p,s) \triangleq \mathbb{E}\left[Q(p)\right| \text{current selling price} = s]$$

is given by

$$Q^c(p,s) = \frac{\partial}{\partial p}\left\{\frac{(\varrho_2(p)-\varrho_1)(1-2\varrho_1+\varrho_1^2+\varrho_1(\varrho_2(p)-\varrho_1))}{(1-\varrho_1)^2(1-\varrho_2(p))}\right\}$$

where $\varrho_2(p) \triangleq \rho F_P(p)$ and $\varrho_1 \triangleq \rho F_P(s)$. Simplifying this expression we obtain

$$Q^c(p,s) = \Big(\frac{1-3\varrho_1+\varrho_1^2+2\varrho_1\varrho_2(p)-\varrho_1\varrho_2^2(p)}{(1-\varrho_1)^2(1-\varrho_2(p))^2}\Big)\rho F_p(p) \tag{4.20}$$

4.3.1 Non-uniform Patience Distribution

Suppose the distribution F_δ of δ is given by

$$F_\delta(x) = \left(\frac{x}{\bar{\bar{\delta}}}\right)^\gamma, \quad 0 \leq \delta \leq \bar{\delta},$$

where $\gamma \in (\frac{1}{2}, 1]$. The ODE describing the CDF $F_P^\gamma(p)$ of equilibrium price distribution is given by

$$\frac{d}{dp}F_P^\gamma(p) = \frac{\mu}{2\bar{\bar{\delta}}}\frac{(1-\rho F_P^\gamma(p))^3}{\rho(1-F_P^\gamma(p))^{\frac{1}{\gamma}}} \tag{4.21}$$

For $\rho = 1$, the equilibrium solution of the ODE (4.21) is given by

$$1 - F_P^{*\gamma}(p) = \frac{1}{\left(1+\frac{(2\gamma-1)\mu}{2\gamma\bar{\bar{\delta}}}p\right)^{\frac{\gamma}{2\gamma-1}}} \tag{4.22}$$

Using (4.19) the expected outstanding inventory is given by

$$Q_\gamma(p) = \frac{\mu}{2\bar{\delta}} \left(1 + \frac{(2\gamma - 1)\mu}{2\gamma\bar{\delta}} p\right)^{\frac{1-\gamma}{2\gamma-1}} \tag{4.23}$$

By substituting $F_P^{*\gamma}$ from (4.22) in (4.14) and solving for p in terms of δ, we obtain that in equilibrium a seller with a patience parameter δ posts the price

$$p_\gamma^*(\delta) = \frac{2\gamma\bar{\delta}}{\mu(2\gamma - 1)} \left(\left(\frac{\bar{\delta}}{\delta}\right)^{(2\gamma-1)} - 1\right) \tag{4.24}$$

At $\gamma = \frac{3}{4}$, we get a power tailed equilibrium selling price distribution as empirically observed in Farmer and Zovko [24] in the context of limit order book. In this case, the outstanding order increases $Q(p) \sim \left(1 + \frac{p}{3}\right)^{\frac{1}{2}}$. Thus, the expected number of outstanding orders between 0 and p, i.e. the market depth,

$$D(p) = \int_0^p Q(p)dp \sim \left(1 + \frac{p}{3}\right)^{\frac{3}{2}}.$$

This market depth implies that the change in price $\frac{dp}{dQ}$ as a result of a market order of Q units is of the order of $Q^{\frac{2}{3}}$, i.e. the price impact $\frac{dp}{dQ}$ is concave in the order quantity.

4.3.2 Summary of Inelastic Markets

The following observations follow form the discussion above.

(a) Everything else being equal, price is higher in less congested (low ρ) markets than in congested (high ρ) markets. This is because as ρ increases, the time to execution at each price level goes up; consequently, the competition between sellers to obtain high prices becomes more intense.

(b) The equilibrium selling prices distribution exhibits power tails. This agrees with the empirical observations (see, e.g. Farmer and Zovko [24]) in limit-book markets where the *market orders* are by definition not sensitive to prices. The market depth $D(p)$ and the price impact function prediction from this simple model agrees with observations in Daniels et al. [17] and Iori et al. [34].

(c) From (4.22) it follows that as the market becomes congested, i.e. $\rho \to 1$, the equilibrium price distribution scales according to $p_s = \frac{\bar{\delta}}{\mu}$. Thus, when the patience parameter $\bar{\delta}$ is held constant, the equilibrium prices are high if μ is low, i.e. the market buy orders appear with a very low frequency; and vice versa.

As $\mu \to \infty$, i.e. the frequency of market buy orders increases, the price scaling $p_s \to 0$, i.e. the effective price window is very small. This has two implications: first, the sellers are not competing on price but on the executing time, and second, the assumption that the buyers are price insensitive is not very serious in practice.

(d) The distribution of the patience parameter F_δ has a significant impact on equilibrium market behavior.

4.4 Multiple Continuous Online Exchange Competing on Quality

In this section, we study an extension to the model presented in § 4.2. Assume that there exist a finite number, L, of exchanges of the form described in § 4.2, indexed by the quality $q \in \{e_1, \ldots, e_L\}$ of the commodity being traded, competing with each other for both sellers and buyers.

On arrival a seller, C^n, based on his beliefs about the expected time to execution at exchange i and price tick j, $T^n(i,j)$ and his patience parameter δ_n decide both the exchange to sell his unit at and the selling price. Thus, sellers strategy $\sigma_n : [0, \bar{\delta}] \mapsto \{1, \ldots, N\} \times \{1, \ldots, L\}$, is to decide the tuple (p_n, q_n), where $p_n \in \{1\epsilon, \ldots, N\epsilon\}$ and $q_n \in \{e_1, \ldots, e_L\}$ as a function of their patience parameter δ_n. Assume that the exogenous dis-utility[3] to the sellers of trading at different exchanges is homogeneous among the seller population and is denoted by $(c_1, c_2, \ldots, c_L)$.

For tractability, we assume that the buyers population is homogeneous in their preference over the set of different quality products available at different prices. Let $\varphi : \{1, \ldots, L\} \times \{1, \ldots, N\} \mapsto \{1, \ldots, NL\}$ denotes the chains which order the set of possible price-quality products according to the homogeneous strict preference order of the buyers. We assume that for all $\forall j < k, i$, we have $\varphi(i,j) < \varphi(i,k)$, i.e. buyers strictly prefers low prices at the same quality and for all $\forall j, k < i$, we have $\varphi(i,j) < \varphi(k,j)$, i.e. buyers strictly prefers high quality at the same price. On arrival, the buyers decides to trades with probability β_{ij} if among the currently available units the *best* (i.e., one with maximum $\varphi(i,j)$) is available at exchange i and price tick j.

Let $X_n(i,j)$ denote the inventory of outstanding orders at price tick j and exchange i when the n-th seller arrives in the exchange. The dynamics described above implies that $\mathbf{X}_n$ is the queue length process for the Markovian preemptive priority queuing system with NL customer classes where the customer class $\varphi(i,j)$ has an arrival rates $\lambda \alpha^{\sigma}_{i,j}$ and a service rate $\mu \beta_{ij}$ where priorities are according to φ and $\alpha^{\sigma}_{i,j}$ are the thinning probabilities under the strategy σ. From Lemma 4.1, it follows that for $i = 1, \ldots, L$ and $j = 1, \ldots, N$ the expected execution time is given by

[3] For example, the production cost of the product of different observable quality being traded at different exchanges or proximity (or ease) of the exchange etc.

$$T^{\alpha}(i,j) = \frac{1}{\mu}\left(\frac{1}{1-\sum_{k=1}^{\phi(i,j)-1}\rho^{\alpha}_{\varphi^{-1}(k)}}\right)\left[\frac{1}{\beta_{ij}} + \frac{\sum_{k=1}^{\varphi(i,j)}(\frac{\rho^{\alpha}_{\varphi^{-1}(k)}}{\beta_{\varphi^{-1}(k)}})}{1-\sum_{k=1}^{\varphi(i,j)}\rho^{\alpha}_{\varphi^{-1}(k)}}\right] \tag{4.25}$$

where

$$\rho^{\alpha}_{ij} \triangleq \frac{\lambda}{\mu}\cdot\frac{\alpha_{ij}}{\beta_{ij}}$$

and λ, μ are the aggregate seller's and buyer's arrival rate.

Thus, using the argument similar to Theorem 4.1, the symmetric stationary BNE selling strategy correspond to the fixed point of the following function.

$$\Psi^{m}_{j}(\alpha) = \mathbb{P}\left[\delta \in \left\{x \in [0,\bar{\delta}] \middle| j\epsilon - xT^{\alpha}(i,j) - c_i \geq k\epsilon - xT^{\alpha}(k,l) - c_k \quad \forall\, k,l\right\}\right] \tag{4.26}$$

In this model, the buyer's demand elasticity with price and quality is two fold. Firstly each buyer prefers a high quality for a given price and low price for a given quality captured by the preferences φ, secondly each buyer independently decides to buy the "best" available object with probability β_{ij} where β_{ij} also decreases with increase in price and decrease in quality. The analytical tractability of the resulting dynamical system mandate such restrictive buyers behavior.

4.4.1 Numerical Example: Multiple Exchanges

Consider a setting with $L = 3$ exchanges and a price grid of $N = 10$ ticks each. Let the tick size is $\epsilon = 1$ unit and the patience parameter δ is distributed uniformly on $[0, 500]$ across the seller population. Suppose the sellers arrive to exchange with an aggregate arrival rate of 10 per unit time. The seller's dis-utility cost, **c** of trading at exchange 1, 2 and 3 is taken to be 7, 3 and 2 respectively.

Suppose the buyers arrival rate is 66 per unit time and the probabilities,

$$\beta_{ij} = \begin{cases} \frac{N-j+2}{11} & i = 1, \\ \frac{7.5+2.5(N-j)}{66} & i = 2, \\ \frac{4+N-j}{33} & i = 3 \end{cases}$$

for all $j = 1, \ldots, N$. The utility to buyers of quality i unit purchased at price j is given by

$$u(i, j) = 30 - 5i - i^2 - 5j$$

where a lower indexed exchange (low i) correspond to high quality. So that the buyers preference chain φ is given by $(1, 1) \to (1, 2) \to (2, 1) \to (1, 3) \to (2, 2) \to \cdots \to (2, 10) \to (3, 8) \to (3, 9) \to (3, 10)$. We compute the equilibrium by solving the optimization problem

$$\alpha^* \in \min_{\alpha \in \mathcal{S}^N} \|\Psi^m(\alpha) - \alpha\|_2 \ , \tag{4.27}$$

where $\Psi^m(\alpha)$ is defined in (4.26).

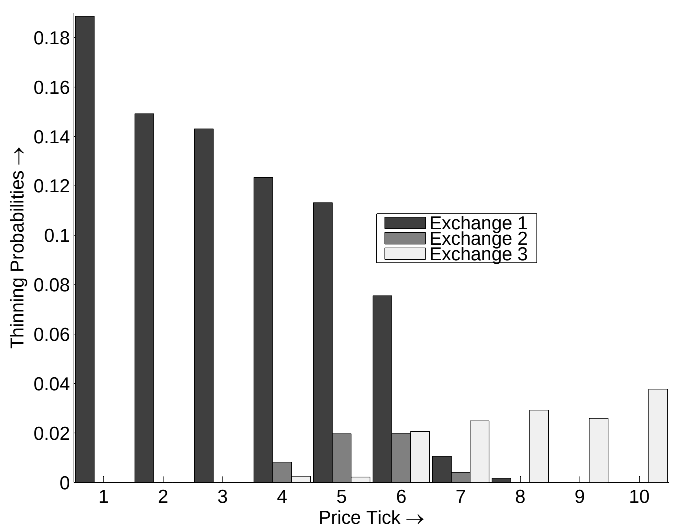


Figure 4.3: The multiple exchange equilibrium solution

The equilibrium thinning probabilities across price ticks for each exchange is plotted in Figure 4.3. The plot is a bar chart with bar heights representing the probabilities. We observe that the equilibrium shows market segmentation along price, quality and cost. Exchange 1, which has high cost (low quality) to the sellers, serves to the low price segment at high frequency, while low quality exchange 3 serves the high price segment at very low frequency. The trades at exchange 2 with intermediate quality occurs at the intermediate prices. Even though the sellers at exchange 1 are incurring high cost and selling at low prices, they are saving considerably in the waiting cost penalty. This is because the demand at low prices and high quality is very high and comes at a high frequency. On the other hand, the rare demand at low quality can only be satisfied after waiting considerably, thus the sellers at exchange 3 sells at high prices.

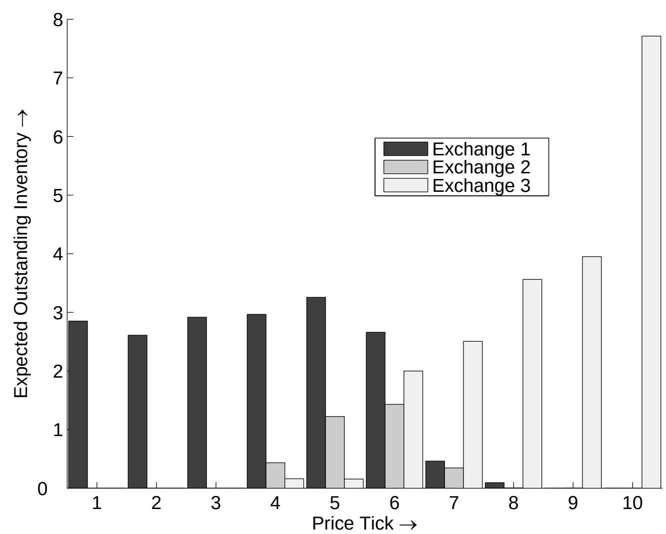


Figure 4.4: Expected outstanding inventory for different exchanges

Figure 4.4 and 4.5 displays the corresponding expected inventory of outstanding units and expected time to execution with price ticks at the three different exchanges.

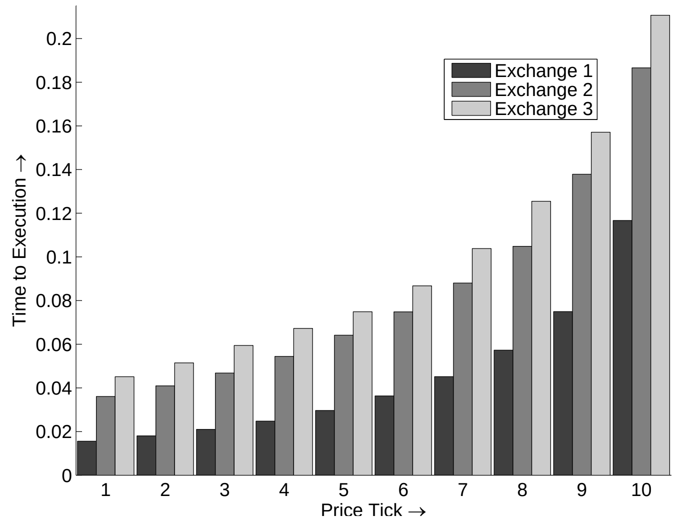


Figure 4.5: Time to execution as a function of price tick at different exchanges

4.5 Conclusion and Extensions

In our model a simple price vs waiting cost trade-off give rise to interesting market dynamics. The equilibrium in our model is numerically computable and can also be solved in closed form in special cases. The model predicts reasonable statistical behaviors of the online exchange market and also agrees with empirical data in some cases. We believe that the equilibrium definition proposed in this chapter can potentially be applied in other dynamic market clearing mechanisms such as,

(a) Modeling order driven financial market where both sides of the market are patient and both buyers and sellers queue up.

(b) Service markets where buyer (sellers) are segmented based on attributes other than price and sellers (buyers) strategically assign resources to each segment.

Appendix A

Preliminaries of Mechanism Design

In economics theory, Mechanism Design refers to the science of designing rules of the game between multiple self interested agents - each with private information about their preference - to achieve centrally desirable outcomes. Many important economic system design problems such as *design of taxation schemes, monopoly pricing, auction design, matching markets and provision of public goods* can be studied in the general framework of Mechanism Design. In recent years, researchers from Computer Science, Operations Research and Operations Management have used this powerful theory to study problems in such inter-disciplinary fields as *electronic markets design, resource allocation* and *decentralized decision making* to give a few examples. In this appendix, we very briefly review the general theoretical framework tailored to its application in chapter 2 and 3. We refer interested reader to Chapter 23 in Mas-Colell et al. [48] - from which this review is primarily taken - for a more detailed introduction to mechanism design. Chapter 2 and 3 in Parkes [56] also provide an excellent survey of mechanism design and recent research directions for its application in e-commerce.

Mechanism design can be described as a three step game of incomplete information, where the agent's type, e.g. marginal cost of production, is private information. In step 1, the principal, e.g. the auctioneer, designs a "Mechanism" or a "Contract", which is basically the rules of the game, e.g. given the bids of the suppliers who supplies how much and at what payments. The agents strategically send some costless message to the principal, e.g. the bids of marginal cost of production, and as a function of those messages the principal select an allocation, e.g. the quantity to be produced by each agent, and the transfer payments to each agent. In step 2, the agents simultaneously decide whether to accept or reject the mechanism and gets some exogenously specified "reservation utility". In step 3, the agents who accepted the mechanism plays the game specified by the mechanism.

Formally, consider a setting with a set $\mathcal{I} \triangleq \{1, \ldots, N\}$ of agents having privately known preference over the outcomes $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^m$. For all $i \in \mathcal{I}$, let $\theta_i \in \Theta_i \subseteq \mathbb{R}^n$ be the private information (aka *type*) of agent i that determine the private preference by the quasi-linear[1] utility function,

$$u_i(\theta, \mathbf{x}, t) \triangleq u_i(\theta, \mathbf{x}) - t_i$$

where $t_i \in \mathbb{R}$ is the transfer payment made by agent i to the principal. Let $\Theta \triangleq \Theta_1 \otimes \Theta_2 \otimes \cdots \otimes \Theta_{|\mathcal{I}|}$ denote the "type space" of possible "type profiles" having a joint prior density $f : \Theta \mapsto \mathbb{R}_{++}$ which is assumed to be common knowledge to all agents and the principal. If the utilities, $u_i(\theta, \mathbf{x})$ does not depend on θ_{-i}, the model is called *private value model*, if in addition, the joint prior distribution is independent across each agent's type space, i.e. $f(\theta) = f_1(\theta_1) \times \cdots \times f_N(\theta_N)$, the model is called *independent private value model*, which is the case in chapter 2 and chapter 3 of this thesis.

Since types are not publicly observable, each agent, simultaneous with other agents is allowed to send a message $m_i \in \mathcal{M}_i$ to the principal. Based on these

[1] A transferable utility function $u : \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}$ is called quasi-linear if there exists a function $U : \mathbb{R}^m \mapsto \mathbb{R}$ such that $u(\mathbf{x}, t) = U(\mathbf{x}) - t$. In this thesis, we primarily used *linear* utility functions, i.e. $u(\mathbf{x}, t) = \theta^{\mathrm{T}}\mathbf{x} - t$.

messages, the principal chose an allocation using the allocation function[2] $\mathbf{x}(.)$ and the transfer payments using the transfer payment function $\mathbf{t}(.)$.

Definition A.1. *A mechanism* $\Gamma = (\mathcal{M}_1 \times \cdots \times \mathcal{M}_N, \mathbf{x}(.), \mathbf{t}(.))$ *is a collection of strategy sets* $(\mathcal{M}_1, \ldots, \mathcal{M}_N)$ *and an allocation function* $\mathbf{x} : M_1 \times \cdots \mathcal{M}_N \mapsto \mathcal{X}$ *and a transfer payment function* $\mathbf{t} : \mathcal{M}_1 \times \cdots \mathcal{M}_N \mapsto \mathbb{R}^N$.

Given a mechanism and the information available, each agents i forms equilibrium bidding strategies $s_i^* : \Theta_i \mapsto \mathcal{M}_i$ under some equilibrium concept which we describe later.

Definition A.2. *A mechanism is said to be* Individually Rational *(IR) at equilibrium* s^* *iff for all* $i \in \mathcal{I}$ *and* $\theta \in \Theta$ *we have* $u_i(\theta, \mathbf{x}(s_1^*(\theta_1), \ldots, s_N^*(\theta_N))) - t_i(s_1^*(\theta_1), \ldots, s_N^*(\theta_N)) \geq \underline{u}_i$ *where* $\underline{u}_i$ *is the reservation utility of agent i.*

Thus Individual Rationality (IR) requires that each agent accept the mechanism in step 2.

Definition A.3. *A mechanism is called direct mechanism iff for all* $i \in \mathcal{I}$, $M_i = \Theta_i$.

Figure A.1 describe the general schematic of the mechanism design framework.

Definition A.4. *A direct mechanism is said to be* incentive compatible *(IC) or truthful iff for all* $i \in \mathcal{I}$ *and* $\theta_i \in \Theta_i$ *we have* $s_i^*(\theta_i) = \theta_i$.

Different notion of equilibrium used to form the equilibrium strategies s_i^* are said to be the equilibrium *solution concepts*. Next we describe the three solution concepts in the framework of truthful direct mechanism starting with the strongest concept of Dominant Strategy equilibrium.

[2]With a slight abuse of notation, we use the same symbol of the allocation function $\mathbf{x}(.)$ (transfer payment function $\mathbf{t}(.)$) and an element in the allocation space $\mathcal{X}$ (payments space $\mathbb{R}^N$).

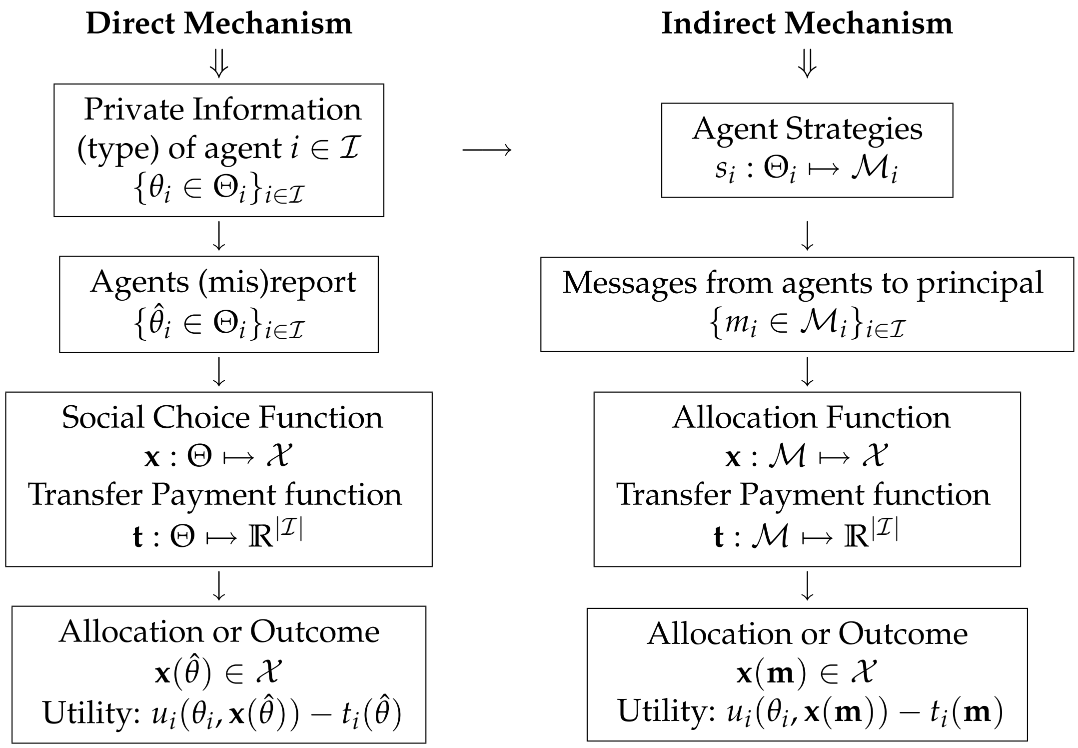


Figure A.1: Mechanism design schematic

Solution Concepts

Definition A.5. *If truth telling is optimal for all agent, for all type profiles and all (mis)reports of other agents, i.e.*

$$u_i((\theta_i, \theta_{-i}), \mathbf{x}((\theta_i, \theta_{-i}), \hat{\theta}_{-i})) - t_i(\theta_i, \hat{\theta}_{-i}) \geq u_i((\theta_i, \theta_{-i}), \mathbf{x}(\hat{\theta}_i, \hat{\theta}_{-i})) - t_i(\hat{\theta}_i, \hat{\theta}_{-i})$$
$$\forall \hat{\theta} \in \Theta, \theta \in \Theta, i \in \mathcal{I} \qquad \text{(Dominant Strategy)}$$

then $(\mathbf{x}, \mathbf{t})$ *is said to be truthful under the dominant strategies equilibrium or dominant strategy incentive compatible.*

Definition A.6. *If truth telling is optimal for all agent, for all type profiles and all truthful reports of other agents, i.e.*

$$u_i((\theta_i,\theta_{-i}),\mathbf{x}(\theta_i,\theta_{-i}))-t_i(\theta) \geq u_i((\theta_i,\theta_{-i}),\mathbf{x}(\hat{\theta}_i,\theta_{-i})) - t_i(\hat{\theta}_i,\theta_{-i})$$
$$\forall\ \hat{\theta}_i \in \Theta_i, \theta \in \Theta, i \in \mathcal{I} \qquad \text{(Ex-post (pointwise) Nash)}$$

then $(\mathbf{x},\mathbf{t})$ *is said to be truthful under ex-post Nash equilibrium or ex-post incentive compatible.*

Definition A.7. *If truth telling is optimal for all agent, for all type profiles conditional on the beliefs about the types of other agents, i.e.*

$$\mathbb{E}_{\theta_{-i}}\left[u_i((\theta_i,\theta_{-i}),\mathbf{x}(\theta_i,\theta_{-i})) - t_i(\theta)\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i((\theta_i,\theta_{-i}),\mathbf{x}(\hat{\theta}_i,\theta_{-i})) - t_i(\hat{\theta}_i,\theta_{-i})\right]$$
$$\forall \theta_i \in \Theta_i, i \in \mathcal{I} \qquad \text{(Ex-ante or BNE)}$$

then $(\mathbf{x},\mathbf{t})$ *is said to be truthful under the Bayesian Nash equilibrium (BNE) or Bayesian incentive compatible.*

The dominant strategy require that truthful bidding is optimal for each agent irrespective of other agent's strategies at all type profiles while ex-post Nash only require that truthful bidding is optimal for each agent if all other agents bid truthfully. It is easy to see that dominant strategy equilibrium is a subset of ex-post Nash equilibrium which is a subset of Bayesian Nash equilibrium. Furthermore, in private value model the dominant strategy solution concept and the ex-post Nash solution concept are equivalent.

Theorem A.1 (Revelation Principle)**.** *Suppose there exists a mechanism* $\Gamma = (\mathcal{M}_1 \times \cdots \times \mathcal{M}_N, \hat{\mathbf{x}}(.), \hat{\mathbf{t}}(.))$ *that results in pointwise allocation* $\mathbf{x}(\theta)$ *and transfer payment* $\mathbf{t}(\theta)$ *at a given Bayesian Nash Equilibrium, then the direct mechanism* $(\Theta, \mathbf{x}(.), \mathbf{t}(.))$ *is Bayesian Nash incentive compatible. The result is also true for Dominant Strategy and Ex-post Nash solution concept.*

Revelation principle (see Harris and Townsend [33] and Myerson [52] among others) is a very general results which allows us to restrict attention to *Incentive Compatible Direct* mechanisms without loss of generality. In much of this thesis we only considered the truthful direct mechanisms, except in § 2.3.3 where we presented an indirect mechanism which implemented[3] the optimal direct mechanism.

Objective function

The principal is assumed to have preference over the set of all incentive compatible and individually rational direct mechanisms. A mechanism design problem is to select a mechanisms based on these preference. Typically, the mechanism, $\Gamma = (\mathbf{x}(.), \mathbf{t}(.))$ is chosen to either maximize ex-ante *efficiency*, i.e. the expected total surplus of all agents and the principal,

$$(\mathbf{x}, \mathbf{t}) \in \operatorname{argmax} \left\{ \int_{\theta \in \Theta} \sum_{i \in \mathcal{I}} u_i(\theta, \mathbf{x}(\theta)) f(\theta) d\theta \right\} \qquad \text{(Expected (ex-ante) efficiency)}$$

or expected revenue to the principal, i.e.

$$(\mathbf{x}, \mathbf{t}) \in \operatorname{argmax} \left\{ \int_{\theta \in \Theta} \sum_{i \in \mathcal{I}} t_i(\theta) f(\theta) d\theta \right\} \qquad \text{(Expected revenue)}$$

Theorem A.2 (Vickrey [68]; Clarke [16]; Groves [31])**.** *In a private value model, suppose the allocation function,*

$$\mathbf{x}^{VCG}(\theta) \in \operatorname{argmax} \left\{ \sum_{i \in \mathcal{I}} u_i(\theta_i, \mathbf{x}) \right\} \tag{A.1}$$

and the transfer payments function,

$$t_i^{VCG}(\theta) = \sum_{j \neq i} u_j(\theta_i, \mathbf{x}^{VCG}(\theta)) - \sum_{j \neq i} u_j(\theta_j, \mathbf{x}_{-i}^{VCG}(\theta_{-i}))$$

[3] An indirect mechanism is said to implement a direct mechanism if it has an equilibrium resulting in the same pointwise allocation and payments as the direct mechanism.

where $\mathbf{x}_{-i}^{VCG}$ is the optimal VCG solution to (A.1) after excluding the agent i, then the mechanism $(\mathbf{x}^{VCG}, \mathbf{t}^{VCG})$ maximize ex-post (hence ex-ante as well) efficiency and is dominant strategy incentive compatible.

Truthful Direct Mechanism Design as an Optimization Problem

Revelation principle implies that, in all its generality, a mechanism design problem is an optimization problem to select a feasible allocation function and a transfer payment function optimizing a given objective subject to incentive compatibility and individual rationality under the chosen solution concept. To given an example, a direct mechanism design problem under dominant strategy solution concept is given by

$$\begin{aligned}
\max_{\mathbf{x}(.),\mathbf{t}(.)} \quad & \mathbb{E}_{\theta\in\Theta}\,\Pi(\theta,\mathbf{x}(\theta),\mathbf{t}(\theta)) \\
\text{s.t.} \quad & \\
\text{IC:} \quad & \theta_i \in \operatorname{argmax}_{\hat{\theta}_i\in\Theta_i}\{u_i(\theta,\mathbf{x}(\hat{\theta}_i,\hat{\theta}_{-i})) - t_i(\hat{\theta}_i,\hat{\theta}_{-i})\} \quad \forall\, i\in\mathcal{I}, \theta_i\in\Theta_i, \hat{\theta}\in\Theta \\
\text{IR:} \quad & u_i(\theta,\mathbf{x}(\theta)) - t_i(\theta) \geq \underline{u}_i \quad \forall\, i\in\mathcal{I}, \theta\in\Theta \\
\text{Feasibility:} \quad & \mathbf{x}(\theta)\in\mathcal{X} \quad \forall\theta\in\Theta
\end{aligned} \tag{A.2}$$

where $\Pi(\theta, \mathbf{x}, \mathbf{t})$ is the objective value at allocation $\mathbf{x} \in \mathcal{X}$, transfer payments $\mathbf{t} \in \mathbb{R}^N$ and type profile $\theta \in \Theta$. We are primarily concerned with expected revenue maximization and the central idea in much of literature as well as this thesis is to be able to simplify (or eliminate) the incentive compatibility and individual rationality constraints to make the mechanism design optimization problem tractable. A celebrated results in that direction is as follows (see it's application in Myerson [53])

Theorem A.3. *Assume that $\mathcal{X} \subseteq \mathbb{R}_+^N$ and for all $i \in \mathcal{I}$,*

1. $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subseteq \mathbb{R}_+$ *(single dimensional type).*

2. $u_i(\theta, \mathbf{x}) \equiv u_i(\theta_i, x_i)$, $u_i \in \mathcal{C}^2$, $u_{i\theta} \triangleq \frac{\partial u_i}{\partial \theta_i} > 0$, $\frac{\partial u_i}{\partial x_i} \geq 0$ *and* $\frac{\partial^2 u_i}{\partial \theta_i x_i} > 0$. *(private value model satisfying "single crossing property")*

then a direct mechanism $(\mathbf{x}(.), \mathbf{t}(.))$ *is dominant strategy incentive compatible if and only if, for all* $i \in \mathcal{I}$ *and* $\theta \in \Theta$,

1. $x_i(\theta_i, \theta_{-i})$ *is non-decreasing in* θ_i.

2. $t_i(\theta) = C_i(\theta_{-i}) + u_i(\theta_i, x_i(\theta)) - \int_{\underline{\theta}_i}^{\theta_i} u_{i\theta}(u, x_i(u, \theta_{-i}))du$

where $C_i : \Theta_{-i} \mapsto \mathbb{R}$ *is some arbitrary function.*

See proposition 23.D.2 in Mas-Colell et al. [48] for proof of a version of the above result. In § 2.3, § 2.4 and § 3.2, we derived similar characterization of incentive compatibility in our models.

Appendix B

Capacitated suppliers with two-dimensional type

In chapter 2 the production cost of a capacitated supplier is of the form

$$c_i(\theta_i, q_i, x) = \begin{cases} \theta_i x, & 0 \le x \le q_i, \\ +\infty, & x > q_i, \end{cases}$$

where the function $c(\cdot)$ is common knowledge but the parameters (θ_i, q_i) are privately known. Thus, the supplier has a two-dimensional type. Alternatively, we can model the cost function c of capacitated suppliers as follows.

$$c(\theta_i, \gamma_i, x) = \theta_i x + \gamma_i x^2,$$

where $\theta_i \in [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$ and $\gamma_i \in [\underline{\gamma}, \bar{\gamma}] \subset \mathbb{R}_+$. In this model, let $\pi_i(\hat{\theta}_i, \hat{\gamma}_i; \theta_i, \gamma_i)$ denotes the surplus when supplier i with type (θ_i, γ_i) bids $(\hat{\theta}_i, \hat{\gamma}_i)$. Since $c(\theta_i, \gamma_i, x)$ is a smooth function, the **IC** conditions simplify to

$$\left.\frac{\partial \pi_i(\hat{\theta}_i, \gamma_i)}{\partial \hat{\theta}_i}\right|_{\hat{\theta}_i = \theta_i} = 0 \qquad \left.\frac{\partial \pi_i(\theta_i, \hat{\gamma}_i)}{\partial \hat{\gamma}_i}\right|_{\hat{\gamma}_i = \gamma_i} = 0 \qquad \forall \theta_i \in (\underline{\theta}, \bar{\theta}), \gamma \in (\underline{\gamma}, \bar{\gamma})$$

These **IC** conditions lead to the following envelope conditions

$$\frac{\partial \pi_i(\theta_i, \gamma_i)}{\partial \theta_i} = -X_i(\theta_i, \gamma_i) \qquad \frac{\partial \pi_i(\theta_i, \gamma_i)}{\partial \gamma_i} = -\mathbb{E}_{(\theta_{-i}, \gamma_{-i})}\, x_i^2(\theta_i, \gamma_i) \tag{B.1}$$

An allocation rule $\mathbf{x}$ is **IC**, i.e. there exists a surplus function π satisfying (B.1) only if

$$\oint_C \begin{pmatrix} -X_i(\theta_i, \gamma_i) \\ -\mathbb{E}_{(\theta_{-i},\gamma_{-i})}\, x_i^2(\theta_i, \gamma_i) \end{pmatrix}^T d\mathbf{l} = 0 \tag{B.2}$$

for all closed smooth paths $C \subset [\underline{\theta}, \bar{\theta}] \times [\underline{\gamma}, \bar{\gamma}]$ and all suppliers i. When (B.2) holds, the surplus $\pi_i(\theta_i, \gamma_i)$ can be written as

$$\pi_i(\theta_i, \gamma_i) = \pi_i(\bar{\theta}, \bar{\gamma}) + \int_l \nabla \pi_i dl \tag{B.3}$$

where $\nabla \pi_i = \begin{pmatrix} -X_i(\theta_i, \gamma_i) \\ -\mathbb{E}_{(\theta_{-i},\gamma_{-i})}\, x_i^2(\theta_i, \gamma_i) \end{pmatrix}$ represent the gradient of the expected surplus function as defined in (B.1) and integration is over *any* path joining $(\bar{\theta}_i, \bar{\gamma}_i)$ and (θ_i, γ_i). Integrating along $(\bar{\theta}_i, \bar{\gamma}_i) \to (\theta_i, \bar{\gamma}_i) \to (\theta_i, \gamma_i)$, we get

$$\pi_i(\theta_i, \gamma_i) = \pi(\bar{\theta}, \bar{\gamma}) + \int_\theta^{\bar{\theta}} X_i(t, \bar{\gamma}) dt + \int_\gamma^{\bar{\gamma}} \mathbb{E}_{(\theta_{-i},\gamma_{-i})}\, x_i^2(\theta_i, t) dt \tag{B.4}$$

Interchanging the order of integrals (assuming some second-order regularity conditions), the buyer's expected profit from any **IC** allocation rule $\mathbf{x}$ can be written as

$$\begin{aligned} \Pi(\mathbf{x}) = \mathbb{E}_{(\theta,\gamma)} \Bigg[& R\left(\sum_{i=1}^n x_i(\theta, \gamma)\right) - \sum_{i=1}^n \left(\theta_i + \frac{F_i(\theta|0)}{f(\theta|0)}\right) x_i(\theta, \gamma) \\ & - \sum_{i=1}^n \left(\gamma_i + \frac{F(\gamma_i|\theta_i)}{f(\gamma_i|\theta_i)}\right) x_i^2(\theta_i, \gamma_i) \Bigg] \end{aligned}$$

Thus, the buyer's optimization problem reduces to selecting a feasible allocation rule $\mathbf{x}$ which maximize the expected profit function $\Pi(\mathbf{x})$ given above subject to the integrability conditions (B.2). This is an optimal control problem and computing its optimal solution is likely to be very hard (see Rochet and Stole [60] and Rochet and Chone [59]).

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