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Cash flow matching: a risk management approach*

Garud Iyengar † Alfred Ka Chun Ma ‡

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Abstract

We propose a scenario based optimization framework for solving the cash flow matching problem where the time horizon of the liabilities is longer than the maturities of available bonds and the interest rates are uncertain. Standard interest rate models can be used for scenarios generation within this framework. The optimal portfolio is found by minimizing the cost at a specific level of shortfall risk measured by the Conditional Tail Expectation (CTE), also known as Conditional Value-at-Risk (CVaR) or Tail-VaR. The resulting optimization problem is still a linear program (LP) as in the classical cash flow matching approach. This framework can be employed in situations when the classical cash flow matching technique is not applicable.

1 Introduction

Bond immunization is a long studied topic initiated by Redington (1952) and Fisher and Weil (1971). Hiller and Schaack (1990) survey various existing approaches to this problem. There are two main techniques namely duration matching and cash flow matching (dedication). A significant limitation of the duration matching approach is that it can only protect against parallel shifts in the yield curve. Fong and Vasicek (1984), Shiu (1987, 1988), Reitano (1996) and Hurlimann (2002) among others have enhanced the duration matching method. In spite of its drawbacks, it is widely used because it is easy to implement.

Another approach is cash flow matching (Kocherlakota et al., 1988, 1990). If the stream of liabilities can be matched perfectly with the asset cash flows, the resulting portfolio is truly immunized to the change of interest rates. However, if the liabilities have a longer time horizon when compared to the maturities of the bonds available in the market, cash flow

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†Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027. Email: garud@ieor.columbia.edu

‡Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027. Email: km2207@columbia.edu

matching does not have a solution. Hiller and Eckstein (1993), Zenios (1995), and Consigli and Dempster (1998) have proposed stochastic programming based approaches for the cash flow matching problem. In this paper, we propose a risk management approach to cash flow matching.

We use the Conditional Tail Expectation (CTE) (Artzner, 1999), also known as Conditional Value-at-Risk (CVaR) (Rockafellar and Uryasev, 2002) or Tail-VaR, to extend the classical cash flow matching technique . CTE is a coherent risk measure (Artzner et al., 1999; Acerbi and Tasche, 2002) and it is interpreted as the conditional expectation above Value-at-Risk (VaR). Instead of requiring all the liabilities to be matched exactly, we compute the minimum cost portfolio constraining the CTE of the maximum shortfall to be non-positive at a prescribed confidence level. Thus, our approach is in the same spirit of Markowitz portfolio selection theory (Markowitz, 1952). As in the classical cash flow matching approach (Kocherlakota et al., 1988, 1990), the resulting scenario based optimization problem remains a Linear Program (LP). The main contribution of this paper is to extend the applicability of the cash flow matching technique while keeping the implementation simple.

2 Model specification

2.1 Notation and assumption

We assume that the cash flows occur at discrete instants of time $t = 0, 1, \ldots, N$. In our model, we attempt to explicitly address the reinvestment risk. To this end, we assume that the same collection of bonds is available for *purchase* at all time $t = 0, 1, \ldots, N$. For example, we could restrict ourselves to a collection of six Treasury bonds with maturities 6 months, 1 year, 2 years, 5 years, 10 years, and 30 years. We will then assume that this collection is available for purchase from the primary market at all time $t = 0, 1, \ldots, N$.

We use the following notation:

$$\begin{aligned}
l_t &= \text{the liability payment at time } t,\\
M &= \text{the number of bonds in our collection,}\\
c_{t,u}^{(j)} &= \text{the cash flow at time } u \text{ from bond } j \text{ purchased at time } t,\\
\mathbf{c}_{t,u} &= \text{the vector } (c_{t,u}^{(1)}, \ldots, c_{t,u}^{(M)})^T,\\
p_t^{(j)} &= \text{the price of bond } j \text{ at time } t,\\
\mathbf{p}_t &= \text{the vector } (p_t^{(1)}, \ldots, p_t^{(M)})^T,\\
x_t^{(j)} &= \text{number of shares of bond } j \text{ purchased at time } t,\\
\mathbf{x}_t &= \text{the vector } (x_t^{(1)}, \ldots, x_t^{(M)})^T,\\
w &= \text{total cost to match the liabilities.}
\end{aligned}$$

We assume that the stream of liabilities l_t is known at time 0. The vectors $\mathbf{c}_{t,u}$ is constant because the collection of bonds is fixed at time 0 and we have $\mathbf{c}_{t,u} = \mathbf{c}_{0,u-t}$ for all $t < u$. The vectors of bond prices $\mathbf{p}_t$ are random variables for $t > 0$. Throughout this paper, we also assume that all bonds are non-callable and default-free.

The vectors $\mathbf{x}_t$, $t = 0, 1, \ldots, N$, and the cost w are the decision variables in the model. The initial bond portfolio is given by $\mathbf{x}_0$. For $t > 0$, $\mathbf{x}_t$ denotes the reinvestment strategy. As in the classical approach, we assume that once we buy the bond we hold it until maturity.

2.2 Classical cash flow matching

Using the notation given in Section 2.1, the classical cash flow matching problem (Kocherlakota et al., 1988, 1990) can be formulated as the following LP:

$$\begin{array}{ll} \min & w \\ \text{s.t.} & w \geq \mathbf{p}_0^T \mathbf{x}_0 + l_0, \\ & l_t - \mathbf{c}_{0,t}^T \mathbf{x}_0 \leq 0, \quad t = 1, \ldots, N, \\ & \mathbf{x}_0 \geq 0. \end{array} \tag{1}$$

The classical cash flow matching problem involves computing the minimum cost portfolio with no shortfall over time, i.e., $l_t - \mathbf{c}_{0,t}^T \mathbf{x}_0 \leq 0$ for all t. However, if the liabilities span a time horizon longer than the maturities of the available bonds, the cash flow matching problem (1) will be infeasible. We consider a risk management approach (Ang and Sherris, 1997) where we reformulate the problem to that of minimizing the cost of the portfolio while controlling the shortfall risk and reinvestment risk at a specified level (Sherris, 1992).

2.3 Extension of the cash flow matching technique using CTE constraints

Kocherlakota et al. (1988) extended the classical technique to allow the cash balance at any time to invest at some fixed rates. Here we address this reinvestment risk by explicitly including the future bond prices to reflect the interest rates for reinvestment. We use the Conditional Tail Expectation to manage the reinvestment risk as well as the shortfall risk.

The Conditional Tail Expectation (CTE) and Value-at-Risk of a random variable Z with cumulative distribution function $F(\cdot)$ with confidence level β are given by

$$\begin{aligned} \mathrm{VaR}_\beta(Z) &= \min_x \{x | F(x) \geq \beta\}, \\ \mathrm{CTE}_\beta(Z) &= \mathbb{E}[Z | Z > \mathrm{VaR}_\beta(Z)], \end{aligned} \tag{2}$$

where $0 < \beta < 1$ and Z represents a loss (see for example Artzner, 1999; Hardy and Wirch, 2004). Rockafellar and Uryasev (2002) show that $\mathrm{CTE}_\beta(Z)$ can be rewritten as

$$\mathrm{CTE}_\beta(Z) = \inf_{\gamma \in \mathbb{R}} \left[\gamma + \frac{1}{1-\beta} \mathbb{E}[Z - \gamma]^+ \right]. \tag{3}$$

In order to exploit this risk measure in the context of the cash flow matching problem, one has to define a reasonable "loss" function. Suppose we denote L_t to be the random variable of shortfall at time t. Then

$$L_t = l_t + \mathbf{p}_t^T \mathbf{x}_t - \sum_{s=0}^{t-1} \mathbf{c}_{s,t}^T \mathbf{x}_s, \quad t = 1, \ldots N. \tag{4}$$

We define the loss associated with a strategy $\{\mathbf{x}_t\}$ to be $\max_{1 \le t \le N} L_t$ and we impose a constraint that $\mathrm{CTE}_\beta(\max_{1 \le t \le N} L_t) \le 0$. Thus, we arrive at the following formulation for the cash flow matching problem:

$$\begin{array}{ll} \min & w \\ \text{s.t.} & l_0 + \mathbf{p}_0^T \mathbf{x}_0 - w \le 0 \\ & \mathrm{CTE}_\beta(\max_{1 \le t \le N} L_t) \le 0, \\ & \mathbf{x}_t \ge 0, \quad t = 0, \ldots, N. \end{array} \tag{5}$$

This formulation reduces to the classical formulation (1) if we force $\mathbf{x}_t = 0$ for $t > 0$ and $\beta > 0$. The rationale for imposing the CTE constraint is to manage the shortfall risk with confidence measured by β. The protection from the shortfall risk increases as the value of β increases. The formulation (5) also manages the reinvestment risk implicit in the reinvestment strategy $\mathbf{x}_t$ due to the uncertainties in the bond prices at time t for $t > 0$.

Although we assume that we buy and hold bonds until maturity, the model can accommodate rebalancing by modifying the definition of L_t. Let $\mathbf{q}_t$ denote the price at time t for the bonds purchased at time $t-1$. Then we define the shortfall as follows:

$$\tilde{L}_t = l_t + \mathbf{p}_t^T \mathbf{x}_t - \mathbf{c}_{t-1,t}^T \mathbf{x}_{t-1} - \mathbf{q}_t^T \mathbf{x}_{t-1}, \quad t = 1, \ldots N, \tag{6}$$

i.e., we sell all the bonds in the very next period to finance the purchase of the new bond portfolio.

2.4 Scenario based optimization with CTE constraints

In general, the optimization problem (5) does not admit an analytical solution. We can, however, employ interest rate models to generate scenarios of future interest rates and bond prices to approximate the expectation involved in computing the CTE.

Let $\{\mathbf{p}_t^j, t = 0, \ldots, N\}$, where $j = 1, \ldots, K$, denote K scenarios for bond prices. Let L_t^j denote the shortfall at time t in the j-th scenario. We then approximate $\mathrm{CTE}_\beta(\max_{1 \le t \le N} L_t)$ by:

$$\min_\gamma \left[\gamma + \frac{1}{K(1-\beta)} \sum_{j=1}^{K} \left(\max_{1 \le t \le N} L_t^j - \gamma \right)^+ \right]. \tag{7}$$

We introduce a new variable $u_j \ge \max_{1 \le t \le N} L_t^j$. Then, it is clear that $\mathrm{CTE}_\beta(\max_{1 \le t \le N} L_t)$

is approximated by the optimal solution of the following LP:

$$\begin{aligned} \min \quad & \gamma + \tfrac{1}{K(1-\beta)} \sum_{j=1}^{K} u_j \\ \text{s.t.} \quad & u_j \geq L_t^j - \gamma, \quad j = 1, \dots, K, \quad t = 1, \dots N, \\ & u_j \geq 0, \quad j = 1, \dots, K, \\ & \gamma \in \mathbb{R}. \end{aligned} \tag{8}$$

The optimal value of the optimization problem (8) is at most zero, if and only if the objective for some feasible solution is non-positive. Thus, it follows that the cash flow matching problem (5) can be approximated by the following LP:

$$\begin{aligned} \min \quad & w \\ \text{s.t.} \quad & l_0 + \mathbf{p}_0^T \mathbf{x}_0 - w \leq 0 \\ & \gamma + \tfrac{1}{K(1-\beta)} \sum_{j=1}^{K} u_j \leq 0 \\ & u_j \geq l_t + (\mathbf{p}_t^j)^T \mathbf{x}_t - \textstyle\sum_{s=0}^{t-1} \mathbf{c}_{s,t}^T \mathbf{x}_s - \gamma, \quad j = 1, \dots, K, \quad t = 1, \dots N, \\ & u_j \geq 0, \quad j = 1, \dots, K, \\ & \gamma \in \mathbb{R}, \\ & \mathbf{x}_t \geq 0, \quad t = 0, \dots, N. \end{aligned} \tag{9}$$

The fact that (9) is an LP makes the implementation of the model in practice simple. This is another advantage of using CTE as the risk measure in our model besides the fact that it is a coherent risk measure.

Note that the choice of interest rate models is flexible. For example, the class of affine models (Duffie and Kan, 1996) which includes popular models like the Vasicek model (Vasicek, 1977) and the Hull-White model (Hull and White, 1990) can be used. In addition, this framework can be easily extended to the case where the stream of liabilities is also random as long as one is able to generate scenarios for the liabilities.

2.5 Solution update

The solution to the LP in (9) is a bond portfolio strategy $\{\mathbf{x}_t \in \mathbb{R}^M : t = 0, \dots, N\}$. If we purchase the bonds as specified by the solution, the shortfall over time is guaranteed to be negative with high probability. On the other hand, as we get more information on the change in interest rates over time, we are able to generate more realistic scenarios and can update the solution accordingly. Suppose at time some time τ, $0 < \tau < N$, we can update the solution $\mathbf{x}_u$ for $u \geq \tau$ by solving a new problem when we face new liabilities

$$\tilde{l}_t = l_t - \sum_{s=0}^{\min(t-1,\tau-1)} \mathbf{c}_{s,t}^T \mathbf{x}_s, \quad t \geq \tau, \tag{10}$$

and the new time horizon is $N - \tau$ instead.

Bond index	Name	Maturity	Coupon rate (%)	Current price
1	T-Bill	6 months	0	95.8561
2	T-Note	1 year	4.5	96.1385
3	T-Note	2 years	4.5	92.6873
4	T-Note	3 years	4.5	89.5784
5	T-Note	4 years	4.5	86.7610
6	T-Note	5 years	4.5	84.1959
7	T-Bond	10 years	5.0	77.5948
8	T-Bond	15 years	5.0	71.9232
9	T-Bond	20 years	5.0	68.1357
10	T-Bond	25 years	5.0	65.5990
11	T-Bond	30 years	5.0	63.8989

Table 1: Details of the Treasury bonds.

3 Numerical example

3.1 Treasury bonds and liabilities

We assume that the collection of bonds available for investment at each time instant is given by the Treasury bonds shown in Table 1. Since the shortest maturity is 6 months, we take 6 mouths as the time step. Recall that $c_{t,u}^{(j)}$ is the cash flow at time u from bond j purchased at time t. For example, for Bond #6 which matures in 5 years, or equivalently 10 time steps, we have

$$c_{t,t+10}^{(6)} = 102.25. \tag{11}$$

The cash flow corresponding to the coupon payments prior to maturity are specified as follows:

$$c_{t,t+1}^{(6)} = \cdots = c_{t,t+9}^{(6)} = 2.25, \tag{12}$$

for $t = 0, \dots N$, and it takes the value zero for all indices not specified above.

We consider the following stream of liabilities:

$$\begin{aligned} l_{2k} &= 100 + k, \quad k = 0, \dots, 10, \\ l_{2k} &= 110 - 2.2 \times (k - 10), \quad k = 11, \dots, 60, \\ l_k &= 0, \quad \text{otherwise}. \end{aligned} \tag{13}$$

Recall that N is the number of time steps and M is number of bonds so $N = 120$ and $M = 11$. Since the time horizon of the liabilities is very long, it is difficult to use the duration matching approach in this case.

3.2 The Hull-White model

We use the Hull-White one-factor model (Hull and White, 1990) for the interest rates. The Hull-White model is an arbitrage-free extension of the Vasicek model (Vasicek, 1977). In this model, the short rate $r(t)$ follows the stochastic differential equation

$$dr(t) = \alpha[\mu(t) - r(t)]dt + \sigma dW(t), \tag{14}$$

where α and σ are constants, $W(t)$ is a standard Brownian motion, and $\mu(t)$ is a deterministic function fitting the initial term structure. The function $\mu(t)$ is given by (see for example Cairns, 2004),

$$\mu(t) = \frac{1}{\alpha}\frac{\partial}{\partial t}F(t) + F(t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-2\alpha t}), \tag{15}$$

where $F(t)$ is the current forward rate. At time t, the price of a zero-coupon bond with face value 1 and maturing at time T is

$$P(t,T) = e^{A(t,T)-B(t,T)r(t)}, \tag{16}$$

where

$$\begin{aligned} B(t,T) &= \frac{1 - e^{-\alpha(T-t)}}{\alpha}, \\ A(t,T) &= \log\frac{P(0,T)}{P(0,t)} + B(t,T)F(t) - \frac{\sigma^2}{4\alpha^3}(1 - e^{-\alpha(T-t)})^2(1 - e^{-2\alpha t}). \end{aligned}$$

The solution to (14) is

$$r(t) = F(t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha t})^2 + \sigma\int_0^t e^{-\alpha(t-s)}dW(s). \tag{17}$$

We assume that the parameters for the Hull-White model are as follows.

$$\begin{aligned} F(t) &= 0.08 + 0.005e^{-0.3t}, \\ \alpha &= 0.24, \\ \sigma &= 0.02. \end{aligned} \tag{18}$$

See Rebonato (1998) for the details of calibration of the Hull-White model. To simulate a scenario of bond prices, we first simulate the short rates $r(0.5)$, ..., $r(60)$ (see for example Glasserman, 2004). Then the simulated bond prices can be found by using (16). For example, the simulated bond prices for Bond #1 are $100\exp(A(0.5,1) - B(0.5,1)r(0.5))$, ..., $100\exp(A(60,60.5) - B(60,60.5)r(60))$.

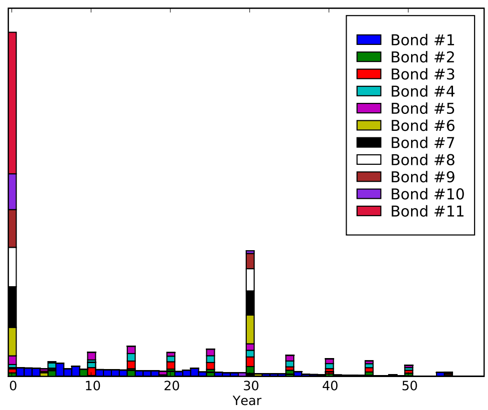
We simulate $K = 1000$ scenarios of interest rates and bond prices for $t = 1, 2, \ldots, N$, and solve the LP in (9) using MOSEK (Andersen and Andersen, 2006).

3.3 Results

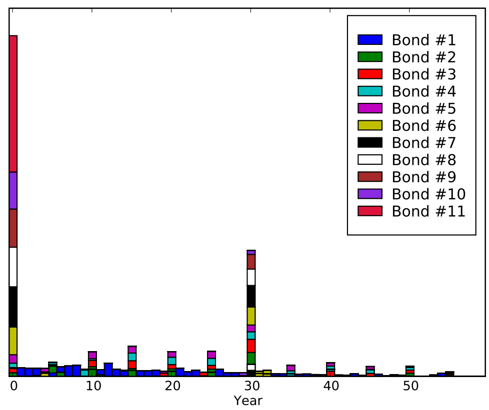
Table 2 shows the bond portfolio at time 0 for different values of β. As the value of β increases, we see the weight on Bond #11 decreases to reduce the reinvestment risk. The cost of the portfolio also increases as we increase the value of β for more protection from the shortfall risk. Figure 1 shows the optimal bond portfolio strategy for $\beta = 0.95$ and $\beta = 0.5$. It illustrates that the reinvestment risk is highest at year 30 where Bond #11 matures.

β	0.9	0.925	0.95	0.975
Bond #1	0.0000	0.0000	0.0000	0.0000
Bond #2	0.1740	0.1735	0.1727	0.1718
Bond #3	0.1948	0.1947	0.1951	0.1936
Bond #4	0.2157	0.2152	0.2146	0.2153
Bond #5	0.2397	0.5093	0.4250	0.2362
Bond #6	1.5765	1.2974	1.3901	1.6066
Bond #7	1.9823	1.9761	1.9879	1.9914
Bond #8	1.8999	1.9268	1.9257	1.9244
Bond #9	1.8400	1.8406	1.8434	1.8491
Bond #10	1.7510	1.7507	1.7502	1.7496
Bond #11	6.9363	6.9283	6.9197	6.8967
Cost	1281.54404	1282.31086	1283.15084	1283.89710

Table 2: Bond portfolio at time 0 for different values of β.



(a) $\beta = 0.95$.



(b) $\beta = 0.5$.

Figure 1: Optimal bond portfolio strategy over time.

4 Conclusions

We revisit the long studied cash flow matching problem and consider it as a portfolio selection problem using CTE as the risk measure. This allows the cash flow matching technique to be used in more general situations. With scenarios generated by any interest rate model, the resulting optimization problem is an LP as in the classical cash flow matching approach. This framework can also be further extended to handle stochastic liabilities without any fundamental change in the algorithm.

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