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Robust Pension Fund Management *

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Abstract

We propose a robust-optimization based model for pension fund management. Our model provides a framework for deciding the amount of contribution to the pension fund that the company should make and how these funds should be allocated. We will show with examples that our model is flexible for decision making in various circumstances.

1 Introduction

Pension plans in the US come in two varieties. Defined *contribution* pension plans specify the *contribution* of the corporation. The employees have the right of investing the corporation's contribution and their own contribution in a limited set of funds. The participants in a defined contributions are responsible for making all the investment decisions and bear all the risks associated with these decisions; thus, the benefits accrued to the participants remains uncertain. In contrast, defined *benefit* pension plans specify the benefits due to plan participants. The plan sponsor, i.e. the corporation, makes all the investment decisions in a defined benefit pension plan and consequently, bears all the investment risk. Defined benefit plans have been in the news in the past few years (see for example WSJ, 2006; Gladwell, 2006) because the pension plans sponsored by many firms are underfunded and some firms face the prospect of declaring bankruptcy over funding pensions. In addition, the recent passage of the Pension Protection Act (HR4, 2006) has added to the urgency of dealing with their crises. While there are many reasons for these funding crises, many of them stemming from demographic reasons, one cannot exclude bad financial planning from this list. It is often cited that the funding crisis began in the early part of this decade because the stock market in 2001 lowered the value of plan holdings and at the same time that interest rates came down resulting in higher values for discounted liabilities. The financial models used to calculate the contribution rate for the firm did not allow for this combination of events; consequently, the pension plans were not in a position to recover from them. There is a need to develop models that account for uncertainty in future market conditions and plan accordingly.

Any useful model for pension fund management must address the following issues.

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(a) How much and when should the plan sponsor contribute to the pension fund?

(b) What portfolio of assets should the fund hold?

(c) What are the projected contributions in the future?

In addition, the model should be able to take taxation policies and other regulatory requirements into account. There are a number of articles in the literature addressing various issues pertinent to pension fund management. Fabozzi et al. (2004, 2005a) discuss the modeling issues arising in the pension fund industry. They conclude that mathematical models significantly improve decision making. Pension fund management is a special case of asset-liability management where the goal is to manage a portfolio of assets to meet known liabilities in the future. See Consigli and Dempster (1998); Klaassen (1998); Drijver et al. (2000); Sodhi (2005) for different approaches to asset-liability management. The typical approach taken by all of these previous approaches is to model the uncertainty in market conditions as random variables but with a known distribution. Then the asset-liability management problem (or, the specific case of pension fund management) can be formulated as a stochastic program. The stochastic program is solved by sampling from the prescribed market distribution and then solving a deterministic optimization problem. As with all sampling-based methods, these approaches suffer from the curse-of-dimensionality and quickly become intractable beyond two-stages. The stochastic programming-based approaches have also been implemented in codes for pension fund management, e.g. Tower Perrin's CAP:Link system (Mulvey et al., 2000).

To model the effect of data uncertainty in optimization problems, Ben-Tal and Nemirovski Ben-Tal and Nemirovski (2001) (see also Ben-Tal et al., 2000) introduced a deterministic framework called *robust optimization*. In this approach, the uncertain parameters are assumed to belong to known and bounded uncertainty sets, and the solution is computed assuming the worst case behavior of the parameters. Typically, the uncertainty sets correspond to confidence regions around point estimates of the parameters; consequently, one is able to provide a probabilistic guarantee on the performance of the solution to the robust problem. For a very large class of uncertainty sets, the robust optimization problem is tractable and scales gracefully with the size of the problem (Ben-Tal and Nemirovski, 2001; Goldfarb and Iyengar, 2003). In this paper we adapt the robust optimization methodology for pension fund management. The following features distinguish our model from those proposed previously in the literature:

(a) We explicitly model the relationship between the corporate structure of the plan's sponsor and the pension fund. In particular, we focus on the effect of corporate contributions to the plan on earnings, leverage and the cost of debt of the sponsoring firm.

(b) Since our model is based on robust optimization methodology, we are able to provide worst case performance guarantees on future contributions.

(c) The computational complexity of the optimization problem that computes the contribution schedule in our model scales gracefully in both the number of assets in the fund's portfolio and the number of trading dates.

(d) Our model can also be used to stress test a given pension fund management strategy, i.e., we are able to compute the worst case contribution rate for a given management strategy.

The rest of this paper is organized as follows. In Section 2, we introduce the proposed robust pension fund management methodology. We discuss a linear approximation that allows us to reformulate the robust optimization problem as a second order cone program (SOCP) in Section 3. In Section 4, we report the results of our numerical experiments. We conclude the paper in Section 5.

2 Pension fund management model

In this section we discuss the model primitives and formulate the optimization problem that computes the optimal contribution schedule and portfolio holdings for the pension fund.

2.1 Assets, liabilities, and dynamics

We assume that the holdings of the pension fund can be rebalanced at discrete instance in time labeled $t = 0, 1, \ldots, N$. For simplicity of exposition, we assume that the universe of assets for the pension funds consists of an equity index and zero-coupon bonds with face value 1 and maturities up to M years. (It will become clear that the set of assets can be increased without altering the model.) Thus, the holdings of the fund at time t can be described by the portfolio vector

$$\boldsymbol{\theta}_t = \begin{bmatrix} \text{Number of shares of 1-year bond} \\ \vdots \\ \text{Number of shares of M-year bond} \\ \text{Number of shares of equity} \end{bmatrix} \in \mathbb{R}^{M+1}.$$

The vector of the asset prices at time t will be denoted by $\mathbf{p}_t$. We denote the contribution of the sponsor to the plan at time t by w_t. Although we implicitly assume that the sponsor has the option of contributing new capital each time the fund holding is rebalanced, we can easily incorporate constraints that limit the time instances when contributions are allowed. From time t to $t+1$, the portfolio vector shifts since all the bonds in the portfolio have a maturity that is one year shorter (the bond with 1-year maturity is now available as cash). Therefore, the value of the portfolio $\boldsymbol{\theta}_t$ at time $t+1$ is given by $\mathbf{p}_{t+1}^T \mathbf{D} \boldsymbol{\theta}_t + \mathbf{d}^T \boldsymbol{\theta}_t$, where

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{d} = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The liability of the pension fund at time t is denoted by l_t. We assume that at time $t = 0$ all the future payments l_t, $t = 0, 1, \ldots, N$ are known *perfectly*, i.e. the uncertainty in the model is only from the changing financial conditions.

2.2 Bond prices and the yield curve

Let $s_{t,j}$ denote the spot risk-free interest rate at time t for a maturity of j years. Following Nelson and Siegel Nelson and Siegel (1987), we assume that

$$s_{t,j} = Z_t^1 + Z_t^2 \left[\frac{1 - \exp(-j/\tau)}{j/\tau} \right] + Z_t^3 \left[\frac{1 - \exp(-j/\tau)}{j/\tau} - \exp(-j/\tau) \right], \tag{1}$$

where the factors Z_t^1, Z_t^2, Z_t^3 refer, respectively, to level, slope, and curvature of the yield curve and τ is a constant. We use the Nelson-Siegel model (1) because it does not result in negative spot rates. This is necessary if we want to discount liabilities with very long duration. The price $B_{t,j}$ at time t of a zero-coupon bond maturing at time $t + j$ is given by

$$B_{t,j} = \frac{1}{(1 + s_{t,j})^j}. \tag{2}$$

We denote the value of the equity index by $q(t)$. We assume that the evolution of the equity index q_t and the factors $\{Z_t^i : i = 1, \ldots, 3\}$ driving the yield curve (1) is given by the stochastic differential equation

$$\begin{bmatrix} dZ_t^1 \\ dZ_t^2 \\ dZ_t^3 \\ \frac{dq_t}{q_t} \end{bmatrix} = \begin{bmatrix} (m_1 - Z_t^1) \\ (m_2 - Z_t^2) \\ (m_3 - Z_t^3) \\ \mu \end{bmatrix} dt + \mathbf{A} d\mathbf{W}_t, \tag{3}$$

where $\mathbf{W}_t = (W_t^1, W_t^2, W_t^3, W_t^4)^T$, $\{W_t^i\}_{t\geq 0}$ are independent standard Brownian motions for $i = 1, 2, 3, 4$, and the lower triangular matrix $\mathbf{A} \in \mathbb{R}^{4\times 4}$ denotes the Cholesky decomposition of the covariance matrix $\mathbf{V} \in \mathbb{R}^{4\times 4}$of the vector $(Z_t^1, \ldots, Z_t^3, q_t)$. The dynamics in (3) imply that each of the factors Z_t^i is an Ornstein-Uhlenbeck process and the equity index q_t is a geometric Brownian motion. The yield curve dynamics given by (3) is similar to the one considered in Fabozzi et al. (2005b). With the above definitions, the price vector is given by

$$\mathbf{p}_t = (B_{t,1}, \ldots, B_{t,M}, q_t)^T.$$

2.3 Financing constraints

We assume that at time 0 we determine the contribution w_t and the portfolio $\boldsymbol{\theta}_t$ for $t = 0, \ldots, \bar{N} \leq N$. Let $\boldsymbol{\psi}$ denote the initial holdings of the fund, i.e. the holdings before re-balancing at time 0. We require that the portfolio $\boldsymbol{\theta}_0$ must satisfy

$$\mathbf{p}_0^T \mathbf{D} \boldsymbol{\psi} + \mathbf{d}^T \boldsymbol{\psi} + w_0 - l_0 = \mathbf{p}_0^T \boldsymbol{\theta}_0, \tag{4}$$

i.e. total value of the portfolio $\boldsymbol{\theta}_0$ must equal the difference between available capital $(\mathbf{p}^T \boldsymbol{\psi} + w_0)$ and the liability l_0. Note that (4) implicitly assumes that the rebalancing does not incur any transaction costs. Therefore, we can without loss of generality assume that the portfolio $\boldsymbol{\psi}$ is all held in cash.

The constraint analogous to (4) for time $t \geq 1$ is

$$\mathbf{p}_t^T \mathbf{D} \boldsymbol{\theta}_{t-1} + \mathbf{d}^T \boldsymbol{\theta}_{t-1} + w_t - l_t = \mathbf{p}_t^T \boldsymbol{\theta}_t.$$

However, conditional on the information available up to time 0, the price vector $\mathbf{p}_t$ is a random variable, i.e. its precise value is *not* known at time 0; therefore, we relax the re-balancing constraint to

$$\mathbb{P}\big(\mathbf{p}_t^T \mathbf{D} \boldsymbol{\theta}_{t-1} + \mathbf{d}^T \boldsymbol{\theta}_{t-1} + w_t - l_t \geq \mathbf{p}_t^T \boldsymbol{\theta}_t\big) \geq 1 - \epsilon_1, \quad t = 1, \ldots, \bar{N} - 1, \tag{5}$$

where ϵ_1 is the probability that the portfolio is infeasible for the liability schedule $\{l_t\}$ and $\mathbb{P}$ denotes the probability measure conditioned on the information available at time 0. The constraints (5) assume that portfolio values $\{\boldsymbol{\theta}_t\}_{t=0}^{\bar{N}-1}$ are all selected with only information available at time 0. In practice, this policy will be rolled out in an online fashion, i.e. at time 0 the policy would be re-balanced and the new portfolio $\boldsymbol{\theta}_0$ held for one period. At time $t = 1$, one would recompute the solution to a new problem that will have a time horizon $\bar{N} + 1$. Thus, $\bar{N}$ denotes the horizon up to which the pension fund wants to plan at time 0.

2.4 Target funding level and regulatory constraints

The constraints (4)-(5) completely ignore the liabilities beyond $\bar{N}$. We expect that liabilities at time $t > \bar{N}$ to have an impact on the portfolio $\boldsymbol{\theta}_0$, although perhaps the effect is not as significant as that of $\{l_t\}_{t=1}^{\bar{N}}$. We model this relative effect through funding level constraints.

Let L_t denote the net present value at time t of the entire set of future liability at a fixed discount rate d, i.e.

$$L_t = \sum_{\tau=t+1}^{N} \frac{l_\tau}{(1+d)^{\tau-t}}.$$

The discount rate d is chosen by the plan sponsor subject to some regulatory constraints. In Section 3.1 we will show that we can set d to a fixed spread over the risk-free rate $s_{t,j}$ without altering the model. The funding level of a pension fund at time t is defined to be the ratio of the total spot value $\mathbf{p}_t^T \boldsymbol{\theta}_t$ of the assets of fund to L_t.

We take the influence of liabilities for $t > \bar{N}$ by setting the set the target funding level at time $\bar{N}$ to a fraction β of the future liabilities, i.e. we require

$$\mathbb{P}(\mathbf{p}_{\bar{N}}^T \mathbf{D} \boldsymbol{\theta}_{\bar{N}-1} + \mathbf{d}^T \boldsymbol{\theta}_{\bar{N}-1} + w_{\bar{N}} \geq l_{\bar{N}} + \beta L_{\bar{N}}) \geq 1 - \epsilon_1. \tag{6}$$

In addition to these constraints, there are other constraints that are necessary to meet various regulatory requirements. For example, in the US, pension funds need to maintain a funding level of $\gamma = 90\%$ and the sponsor is required to contribute if the funding level drops below γ. Constraints of this kind can be imposed by adding constraints of the form:

$$\mathbf{p}_0^T \boldsymbol{\theta}_0 \geq \gamma L_0, \tag{7}$$

and

$$\mathbb{P}(\mathbf{p}_t^T \boldsymbol{\theta}_t \geq \gamma L_t) \geq 1 - \epsilon_2, \quad t = 1, \ldots, \bar{N} - 1, \tag{8}$$

where ϵ_2 is the probability that the portfolio is below the funding level γ for $t = 1, \ldots, \bar{N} - 1$. See Fabozzi et al. (2004) for a summary of regulations on pension funds in different countries.

2.5 Objective

Clearly, one goal of the plan's sponsor is to minimize the cost of funding the pension fund. Thus, one possible objective could be to minimize the net present value of all the contributions, i.e.

$$\min \quad \sum_{t=0}^{\bar{N}} w_t B_{0,t}. \tag{9}$$

However, the objective (9) does not adequately represent the cost to the sponsoring firm of its contributions. Moreover, (9) completely ignores the effect of taxes.

Following the Meyers and Majluf pecking order hypothesis (Myers and Majluf, 1984), we assume that the sponsor would first use earnings, and next use debt to finance its contributions $\{w_t\}$. We assume that the firm will not be able to issue equity for the purpose of meeting its pension obligations. Suppose w_t^e denotes the portion of the pension fund contribution w_t that is financed directly from the firm's earnings C_t before interest and tax (EBIT). We assume that C_0 is known and the portion of the earnings invested in the firm grows at a rate r_e. Thus,

$$C_{t+1} = (C_t - w_t^e)(1 + r_e). \tag{10}$$

We also impose the additional constraint $w_t^e \leq u C_t$, where $u \in [0, 1]$. Note that r_e is *not* the cost of equity for the firm. This is rate at which earning grow and may, in fact, be smaller than the cost of equity implied by the CAPM.

Let T be the marginal tax rate. Suppose w_t^d is the amount raised in the debt market at time t. We will assume that this debt has to be returned at the end of $D = 1$ period. (It will be clear that the theory extends easily to handle the case $D \neq 1$.) Since the interest payments are tax deductible, the cost incurred on this debt issue is

$$\begin{aligned} c_t^d(P) &= \left(\frac{1+(1-T)(s_{t,1}+P)}{1+s_{t,1}}\right) B_{0,t} w_t^d \\ &= \Big((1-T)B_{0,t} + ((1-T)P+T)B_{0,t+1}\Big) w_t^d, \end{aligned}$$

where P is the spread paid by the firm over the risk-free rate. Note that $(1-T)(s_{t,1}+P)$ denotes the firm's cost of debt.

The spread P is a function of the debt rating of the firm. We assume that the spread is a function of the interest coverage (IC_t). Since we assume that each debt offering has a duration $D = 1$, it follows that the interest coverage IC_t is given by

$$IC_t = \frac{C_t}{(s_{t,1}+P)w_t^d}.$$

The firm has a particular debt rating Q provided $IC_t \in [\alpha(Q), \beta(Q)]$ and in this case the spread is given by $P(Q)$ (Damodaran, 2004). We assume that the firm has to always maintain a debt rating $Q \geq \underline{Q}$, i.e.

$$\alpha(\underline{Q})\Big(s_{t,1} + P(\underline{Q})\Big) w_t^d \leq C_t, \tag{11}$$

and approximate the cost of debt at time t by $C_t^d(P(\underline{Q}))$. Since $s_{t,1}$ in (11) is not known at time 0 for $t > 0$, we relax (11) to

$$\mathbb{P}\left(\alpha(\underline{Q})\Big(s_{t,1} + P(\underline{Q})\Big) w_t^d \leq C_t\right) \geq 1 - \epsilon_3, \tag{12}$$

where $t = 1, 2, \ldots, \bar{N}$. Note that $C_t^d(P(\underline{Q}))$ overestimates the cost of debt because it ignores the fact that when w_t^d is small the spread $P < P(\underline{Q})$. From (10), it follows that financing the contribution w_t from the earnings C_t affects both the ability of using future earning to fund pension requirements as the cost of debt.

We model the impact of cash and debt financing by setting the objective of the portfolio selection problem to

$$\min \quad \sum_{t=0}^{\bar{N}} C_t^d(P(\underline{Q})). \tag{13}$$

Collecting together all the constraints, it follows that the pension fund management problem can be formulated as the following optimization problem

$$\begin{array}{rl} \min & \sum_{t=0}^{\bar{N}} \Big((1-T)B_{0,t} + ((1-T)P(\underline{Q})+T)B_{0,t+1}\Big) w_t^d \\ \text{subject to} & (4),\ (5),\ (6),\ (7),\ (8),\ (10) \text{ and } (12). \end{array} \tag{14}$$

3 Linearization and second-order cone programming

The constraints (5), (6), (8) and (12) are all *nonlinear* chance constraints. The non-linearity in these constraints arises from the fact that the bond prices $B(t,j)$ are nonlinear functions of the factors (Z_1, Z_2, Z_3, q). Nonlinearity does not directly affect the sampling-based methods – one can still generate samples of the factors (Z_1, Z_2, Z_3, q) and then solve a deterministic nonlinear program to approximate the stochastic problem. However, the nonlinearity does affect the convergence of the solution of the deterministic approximation to the true solution.

In this paper, we take a different approach. Note that the interval between rebalancing is under the control of the portfolio manager and is likely to be inversely proportional to the volatility of the assets. Therefore, the portfolio value will be close to its expected value and the contribution of volatility is likely to be small. Motivated by this observation, we *linearize* the nonlinear dynamics. Recall that in any practical implementation the solution will be "rolled out" in an online fashion, i.e. the solution will be recomputed after one step, therefore, the error induced by linearization is likely to be small. The linearization allows us to formulate chance constraints as deterministic robust constraints. We will comment on the quality of the linear approximation in Section 4.

3.1 Linearization

The probabilistic constraints are approximated via linearizing the bond price using Itô-Taylor expansion, (see Kloeden and Platen, 1999, for example). The Itô-Taylor expansion applied to (2) at time 0 using (1) and (3) implies that

$$\begin{aligned} B_{t,j} \approx\ & B_{0,j} + \left(\sum_{i=1}^{3} (m_i - Z_0^i) \left.\frac{\partial B_{s,j}}{\partial Z_s^i}\right|_{s=0} + \frac{1}{2} \sum_{i,l=1}^{3} \rho_{il} \left.\frac{\partial^2 B_{s,j}}{\partial (Z_s^i) \partial (Z_s^l)}\right|_{s=0} \right) t \\ & + \sum_{i=1}^{3} \left.\frac{\partial B_{s,j}}{\partial Z_s^i}\right|_{s=0} \sum_{k=1}^{4} v_{ik} W_t^k, \end{aligned} \tag{15}$$

where $\rho_{il} = \sum_{k=1}^{4} v_{ik} v_{kl}$ and $\mathbf{A} = [v_{ij}]$ is the covariance matrix of the factor vector $(W_t^1, \ldots, W_t^4)$. Similarly, for the equity index q_t, we have that

$$q_t \approx q_0 + q_0 \mu t + q_0 \sum_{k=1}^{4} v_{4k} W_t^k. \tag{16}$$

Thus,

$$\mathbf{p}_t \approx \mathbf{p}_0 + \delta \mathbf{p}_0 t + \nabla \mathbf{p}_0 \mathbf{A} \mathbf{W}_t, \tag{17}$$

where

$$\delta \mathbf{p}_0 = \begin{bmatrix} \sum_{i=1}^{3} (m_i - Z_0^i) \left.\frac{\partial B_{s,1}}{\partial Z_s^i}\right|_{s=0} + \frac{1}{2} \sum_{i,l=1}^{3} \rho_{il} \left.\frac{\partial^2 B_{s,1}}{\partial (Z_s^i) \partial (Z_s^l)}\right|_{s=0} \\ \vdots \\ \sum_{i=1}^{3} (m_i - Z_0^i) \left.\frac{\partial B_{s,M}}{\partial Z_s^i}\right|_{s=0} + \frac{1}{2} \sum_{i,l=1}^{3} \rho_{il} \left.\frac{\partial^2 B_{s,M}}{\partial (Z_s^i) \partial (Z_s^l)}\right|_{s=0} \\ q_0 \mu \end{bmatrix} \in \mathbb{R}^{M+1},$$

and

$$\nabla \mathbf{p}_0 = \begin{bmatrix} \left.\frac{\partial B_{s,1}}{\partial Z_s^1}\right|_{s=0} & \left.\frac{\partial B_{s,1}}{\partial Z_s^2}\right|_{s=0} & \left.\frac{\partial B_{s,1}}{\partial Z_s^3}\right|_{s=0} & \left.\frac{\partial B_{s,1}}{\partial q_s}\right|_{s=0} \\ \vdots & \vdots & \vdots & \vdots \\ \left.\frac{\partial B_{s,M}}{\partial Z_s^1}\right|_{s=0} & \left.\frac{\partial B_{s,M}}{\partial Z_s^2}\right|_{s=0} & \left.\frac{\partial B_{s,M}}{\partial Z_s^3}\right|_{s=0} & \left.\frac{\partial B_{s,M}}{\partial q_s}\right|_{s=0} \\ 0 & 0 & 0 & q_0 \end{bmatrix} \in \mathbb{R}^{(M+1)\times 4}.$$

3.2 The Second Order Cone programming

From (17), it follows that for all $t = 1, \ldots, \bar{N} - 1$,

$$\begin{aligned} & \mathbb{P}(\mathbf{p}_t^T \mathbf{D} \boldsymbol{\theta}_{t-1} + \mathbf{d}^T \boldsymbol{\theta}_{t-1} + w_t - l_t \geq \mathbf{p}_t^T \boldsymbol{\theta}_t) \\ & \quad = \mathbb{P}(\mathbf{p}_t^T (\mathbf{D} \boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t) + \mathbf{d}^T \boldsymbol{\theta}_{t-1} + w_t - l_t \geq 0) \\ & \quad = \mathbb{P}\Big((\mathbf{p}_0 + \delta \mathbf{p}_0 t)^T (\mathbf{D} \boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t) + \mathbf{d}^T \boldsymbol{\theta}_{t-1} + w_t - l_t \geq -(\mathbf{D} \boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t)^T \nabla \mathbf{p}_0 \mathbf{A} \mathbf{W}_t \Big). \end{aligned}$$

Since $-(\mathbf{D}\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t)^T \nabla \mathbf{p}_0 \mathbf{A} \mathbf{W}_t \sim N(0, \|(\mathbf{D}\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t)^T \nabla \mathbf{p}_0 \mathbf{A}\|^2 t)$, where $\|\cdot\|$ denotes the Euclidean norm, and $\epsilon_1 < 0.5$, we have that

$$\begin{gathered} \mathbb{P}(\mathbf{p}_t^T \mathbf{D}\boldsymbol{\theta}_{t-1} + \mathbf{d}^T \boldsymbol{\theta}_{t-1} + w_t - l_t \geq \mathbf{p}_t^T \boldsymbol{\theta}_t) \geq 1 - \epsilon_1 \\ \Updownarrow \\ (\mathbf{p}_0 + \delta \mathbf{p}_0 t)^T (\mathbf{D}\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t) + \mathbf{d}^T \boldsymbol{\theta}_{t-1} + w_t - l_t \geq \sqrt{t} \Phi^{-1}(1 - \epsilon_1) \|(\mathbf{D}\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t)^T \nabla \mathbf{p}_0 \mathbf{A}\|, \end{gathered} \tag{18}$$

where $\Phi(\cdot)$ denotes cumulative density function of the standard normal random variable. The constraint (18) is of the form

$$\|\mathbf{B}\mathbf{x} - \mathbf{a}\| \leq \mathbf{d}^T \mathbf{x} + c,$$

where $\mathbf{B}$, $\mathbf{a}$, $\mathbf{d}$ and c are constants and $\mathbf{x}$ is the decision variable. Constraints of this form are called *second-order cone* (SOC) constraints.

Using an analysis similar to the one employed above, (6) is can be reformulated as the SOC constraint

$$(\mathbf{p}_0 + \delta \mathbf{p}_0 \bar{N})^T \mathbf{D}\boldsymbol{\theta}_{\bar{N}-1} + \mathbf{d}^T \boldsymbol{\theta}_{\bar{N}-1} + w_{\bar{N}} - l_{\bar{N}} - \beta L_{\bar{N}} \geq \sqrt{\bar{N}} \Phi^{-1}(1 - \epsilon_1) \|(\mathbf{D}\boldsymbol{\theta}_{\bar{N}-1})^T \nabla \mathbf{p}_0 \mathbf{A}\|, \tag{19}$$

and the regulation constraint (8) reformulated as the SOC constraint

$$(\mathbf{p}_0 + \delta \mathbf{p}_0 t)^T \boldsymbol{\theta}_t - \gamma L_t \geq \Phi^{-1}(1 - \epsilon_2) \sqrt{t} \|\boldsymbol{\theta}_t^T \nabla \mathbf{p}_0 \mathbf{A}\|. \tag{20}$$

Since the short rates $s_{t,1}$ are described by a Ornstein-Uhlenbeck process whose marginal distribution is normal, it follows that the interest-coverage constraint (12) is equivalent to the linear constraint

$$\alpha(\underline{Q}) \Big(\mathbb{E}(s_{t,1}) + \sqrt{\mathbf{var}\,[s_{t,1}]} \Phi(1 - \epsilon_3) + P(\underline{Q}) \Big) w_t^d \leq C_t, \qquad t = 1, 2, \ldots, \bar{N}, \tag{21}$$

where $\mathbf{var}\,[s_{t,1}]$ denotes the variance of $s_{t,1}$.

Thus, the linear approximation introduced in Section 3.1 implies that the pension fund management problem (14) can be reformulated as

$$\begin{array}{rl} \min & \sum_{t=0}^{\bar{N}} \Big((1-T) B_{0,t} + \Big((1-T) P(\underline{Q}) + T \Big) B_{0,t+1} \Big) w_t^d \\ \text{subject to} & (4),\ (18),\ (19),\ (7),\ (20),\ (10) \text{ and } (21). \end{array} \tag{22}$$

This optimization problem has a linear objective with linear and SOC constraints. A problem with this structure is called a *second-order cone program* (SOCP). The fact that the pension fund management problem can be approximated by an SOCP implies that very large scale problems can be solved efficiently both in theory (Alizadeh and Goldfarb, 2003) and in practice (Andersen and Andersen, 2006). Moreover, a number of commercial solvers such as MOSEK, CPLEX and Frontline System (supplier of EXCEL SOLVER) provide the capability for solving SOCPs in a numerically robust manner.

4 Numerical Examples

In this section we report the results of our numerical experiments with the linearized pension fund management problem (22).

4.1 Parameters

We can chose the S&P500 index as the equity index and estimate the parameters defining the dynamics of the yield curve using yearly historical data for the 3-month, 6-month, 2-year, 3-year, 5-year, 10-year, and 30-year interest rates.

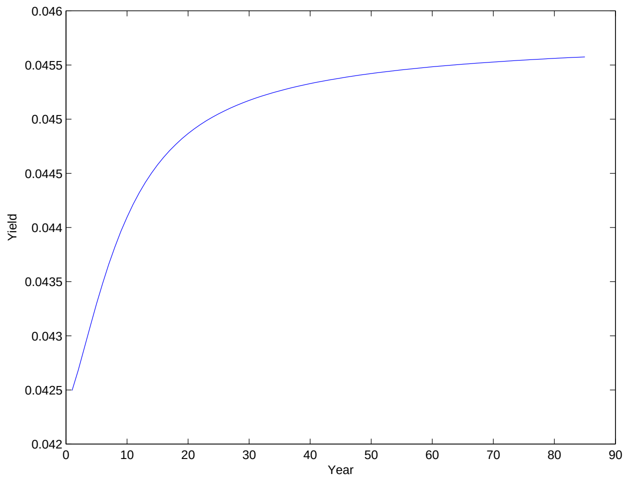

Figure 1: Current Yield Curve

Following Fabozzi et al. (2005c) (see also Barrett et al., 1995), we set $\tau = 3$. The other parameters used in the example are:

$$Z_0^1 = 4.5794, \quad Z_0^2 = -0.3443, \quad Z_0^3 = -0.2767, \quad q_0 = 1248.29,$$

$$\mu = 0.0783, \quad m_1 = 6.1694, \quad m_2 = -2.4183, \quad m_3 = 0.4244,$$

the covariance matrix

$$\mathbf{V} = \begin{bmatrix} 2.1775 & -4.5778 & 19.3399 & -0.1201 \\ -4.5778 & 15.6181 & -43.6039 & 0.2679 \\ 19.3399 & -43.6039 & 179.7153 & -1.0094 \\ -0.1201 & 0.2679 & -1.0094 & 0.0078 \end{bmatrix},$$

and the correlation matrix

$$\boldsymbol{\rho} = \begin{bmatrix} 1.0000 & -0.2178 & 0.5685 & -0.4008 \\ -0.2178 & 1.0000 & -0.4452 & 0.0945 \\ 0.5685 & -0.4452 & 1.0000 & -0.1149 \\ -0.4008 & 0.0945 & -0.1149 & 1.0000 \end{bmatrix}.$$

Thus, the Cholesky decomposition $\mathbf{A}$ of $\mathbf{V}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1.4756 & 0 & 0 & 0 \\ -3.1023 & 2.4482 & 0 & 0 \\ 13.1063 & -1.2027 & 2.5485 & 0 \\ -0.0814 & 0.0063 & 0.0255 & 0.0212 \end{bmatrix}.$$

These parameter estimates result in the current yield curve shown in Figure 1. The number of maturities M was set to $M = 10$ in our numerical experiments.

The liability stream that was used in our numerical experiments is shown in Figure 2. The liability stream ends in year $N = 85$.

The parameters ϵ_j, $j = 1, 2, 3$, that control the probability of violation were selected by the Bonferroni procedure. For example, if we require that all the feasibility inequities be satisfied with probability p we set $\epsilon_1 = \frac{1-p}{N}$.

Using these parameters we solved the linearized pension fund management problem (22). The output of this optimization problem consisted of

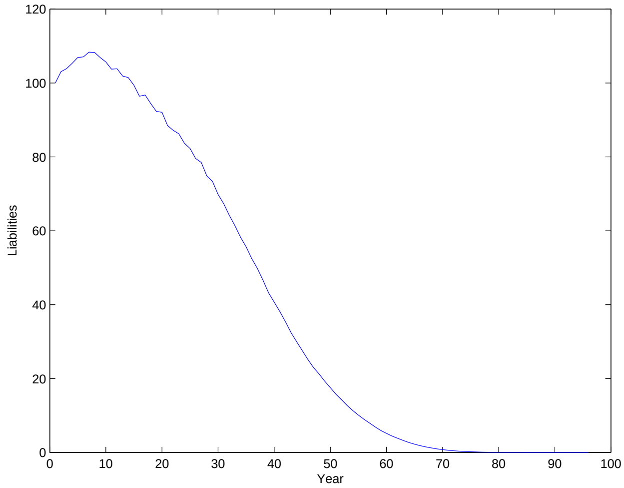

Figure 2: Liability as function of time

- the contribution financed from earnings: w_t^e, $t = 0, 1, \ldots, \bar{N}$,
- the contribution financed from issuing debt: w_t^d, $t = 0, 1, \ldots, \bar{N}$, and
- the portfolio holdings: $\boldsymbol{\theta}_t$, $t = 0, 1, \ldots, \bar{N} - 1$.

4.2 Selecting the optimal pension fund management strategy

In this section, we use the linearized pension fund management problem (22) to solve for the optimal contribution plan and portfolio holding for a plan with the value of initial holding

$$\mathbf{p}_0^T \boldsymbol{\psi} = 0.8(L_0 + l_0),$$

Other parameters for this numerical example are set as follows:

(i) We considered the optimal plan for the first 4 years, i.e. $\bar{N} = 4$.

(ii) The regulation mandated minimum funding level γ was set to $\gamma = 0.9$,

(iii) the target funding level β that controls the influence of liabilities beyond $\bar{N}$ was set to $\beta = 0.9$,

(iv) the liabilities were discounted at a discount rate d of 0.06.

We set the probability that any of the chance constraints is violated to $1 - p = 0.01$, i.e., we allowed for a 1% chance that one of the chance constraints will be violated.

We assumed that the earnings $C_0 = 500$ and set $u = 0.2$, i.e., we imposed a limit that at most 20% of the earnings can be used to fund the pension plan. The marginal tax rate T was set to $T = 0.35$ and we assumed that the company wants to maintain a credit rating $Q =$ "A+", i.e. $\alpha(\underline{Q}) = 5.5$ and $P(\underline{Q}) = 0.008$ (Damodaran, 2004).

The optimal contribution plans as a function of the growth rate of revenue r_e are displayed in Table 1 In this table, w_0^d denotes the amount of contribution at time 0 that is financed by debt, w_0^e denotes the amount that is financed by internal earnings, `Equity value` refers to the total dollar value of the equity held in the optimal portfolio at time 0, `total value` denotes the total value of the holdings of the portfolio, `equity ratio` denotes the ratio of equity value to the total value, `optimal value` denotes the optimal objective value and $\sum_{t=0}^{\bar{N}} B_{0,t} w_t$ is the total discounted value of the contributions over the next $\bar{N} = 4$ years. Since we expect that the solution of the plan will be implemented in an online manner, only the solution for time 0 is displayed in Table 1. The results support the following observations:

r_e	0.000	0.025	0.050	0.075	0.100
w_0^d	214.10	208.35	202.47	196.46	190.32
w_0^e	100.00	100.00	100.00	100.00	100.00
Equity Value	321.34	323.72	320.47	316.67	312.72
Total Value	1455.07	1449.33	1443.45	1437.44	1431.29
Equity Ratio	0.22	0.22	0.22	0.22	0.22
Optimal Value	212.11	206.42	200.59	194.64	188.55
$\sum_{t=0}^{\bar{N}} B_{0,t} w_t$	512.51	517.00	521.76	526.80	532.11

Table 1: Optimal pension plan as a function of revenue growth rate r_e ($C_0 = 500$)

C_0	10.000	250.000	500.000	750.000	1000.000
w_0^d	10.32	180.03	202.47	78.63	0.00
w_0^e	-11.82	50.00	100.00	150.00	198.06
Equity Value	8479.89	275.86	320.47	282.08	193.16
Total Value	8735.55	1371.01	1443.45	1369.61	1339.04
Equity Ratio	0.97	0.20	0.22	0.21	0.14
Optimal Value	138.01	324.91	200.59	77.90	0.00
$\sum_{t=0}^{\bar{N}} B_{0,t} w_t$	146.40	487.50	521.76	557.92	632.53

Table 2: Optimal pension plan as a function of initial revenue C_0 ($r_e = 0.05$)

(a) It is always cheaper to use operating revenues to fund the pension obligations.

(b) The optimal equity ratio at time 0 appears to be independent of the growth rate r_e. Since equity has a higher return, one would have expected the equity ratio of the optimal pension plan to increase with a decrease in the return r_e.

(c) The objective value of (22) and the amount of debt financing decrease with the increase in growth rate r_e. Since the objective in (22) only considers the cost of debt financing, both of these observations reiterate the fact that operating revenues are the cheaper source of revenue.

Table 2 displays the results for optimal contribution rate as a function of the initial capital C_0 for a fixed growth rate $r_e = 0.05$. The following observations follow from the results.

(a) The pension fund management problem is infeasible for $C_0 = 10$, i.e., the sponsor cannot both fund the pension fund and keep its credit rating $Q =$ "A+".

(b) The optimal objective value decreases as C_0 increases despite that w_0^d is not decreasing from $C_0 = 250$ to $C_0 = 500$.

(c) When the operating income is very high ($C_0 = 1000$), the objective is zero since the pension fund obligations can be completely funded from operating revenues.

(d) The total discounted payment $\sum_{t=o}^{\bar{N}} B_{0,t} w_t$ *increases* as the operating revenue *increases*. This is surprising, although it does not contradict any of the modeling assumption. The robust problem minimizes the cost of borrowing with the constraint that credit rating of the company should not suffer – this may result in higher discounted payments.

Tables 3 and 4 show the results when one removes the regulation constraints. Note that we only remove constraints (7) and (8) but the keep the target funding level constraint (6).

r_e	0.000	0.025	0.050	0.075	0.100
w_0^d	0.00	0.00	0.00	0.00	0.00
w_0^e	93.48	92.21	92.13	90.36	88.80
Equity Value	92.46	84.07	120.87	125.36	143.20
Total Value	1234.46	1233.19	1233.11	1231.33	1229.78
Equity Ratio	0.07	0.07	0.10	0.10	0.12
Optimal Value	0.00	0.00	0.00	0.00	0.00
$\sum_{t=0}^{\bar{N}} B_{0,t} w_t$	299.70	309.93	322.10	333.46	345.75

Table 3: Optimal pension plan as a function of r_e ($C_0 = 500$ and no-regulation)

C_0	10.000	250.000	500.000	750.000	1000.000
w_0^d	7.61	0.00	0.00	0.00	0.00
w_0^e	-5.51	49.69	92.13	133.07	176.44
Equity Value	2914.79	2.99	120.87	368.23	484.84
Total Value	3690.45	1190.66	1233.11	1274.04	1317.42
Equity Ratio	0.79	0.00	0.10	0.29	0.37
Optimal Value	123.64	0.00	0.00	0.00	0.00
$\sum_{t=0}^{\bar{N}} B_{0,t} w_t$	126.43	169.22	322.10	475.60	614.76

Table 4: Optimal pension plan as a function of initial revenue C_0 ($r_e = 0.05$ and no regulation)

Thus, these results can be interpreted as the optimal decision for a corporation that is given up to $\bar{N} = 4$ years to reach the target of $\gamma = 0.9$. The first thing we notice is that both the optimal objective value and the total discounted payments significantly decrease. This leads us to conclude that the regulation requirements are the main binding constraints. Another significant change is that the equity ratio is no longer flat as a function of r_e or C_0 – the optimal plan invests more in equity as r_e (or C_0) increases. One possible explanation for this is that as r_e increases the sponsor is able to reduce payments by investing in equity and hedge the higher risk by having a larger revenue cushion. It appears that the optimal plan with regulatory constraints is not investing in equity because it is not able to hedge the increased risk.

A graphical summary of contribution and optimal value for different cases is given in the Figure 3.

Tables 5 and 6, respectively, display the optimal contribution with and without regulation as a function of the initial funding level, i.e. $\mathbf{p}_0^T \boldsymbol{\psi}/L_0$, with $r_e = 0.05$ and $c_0 = 500$. These results are consistent with the other results in this section. The optimal equity-ratio with and without regulation increases with the increase in the funding level – the rate of increase is higher when there are no regulatory constraints. Note, however, that the equity ratio does not increase as the initial funding level decreases. We believe that this is a consequence of the debt level constraints.

We also experimented with the bound u and found that for small r_e and small C_0, the optimal plan tends to use up the entire operating revenue. This is a consequence of the fact that our model does not consider the entire corporate strategy of the sponsor. Therefore, it is imperative that the sponsor selects u carefully.

A graphical summary of equity ratio for different cases is given in the Figure 4.

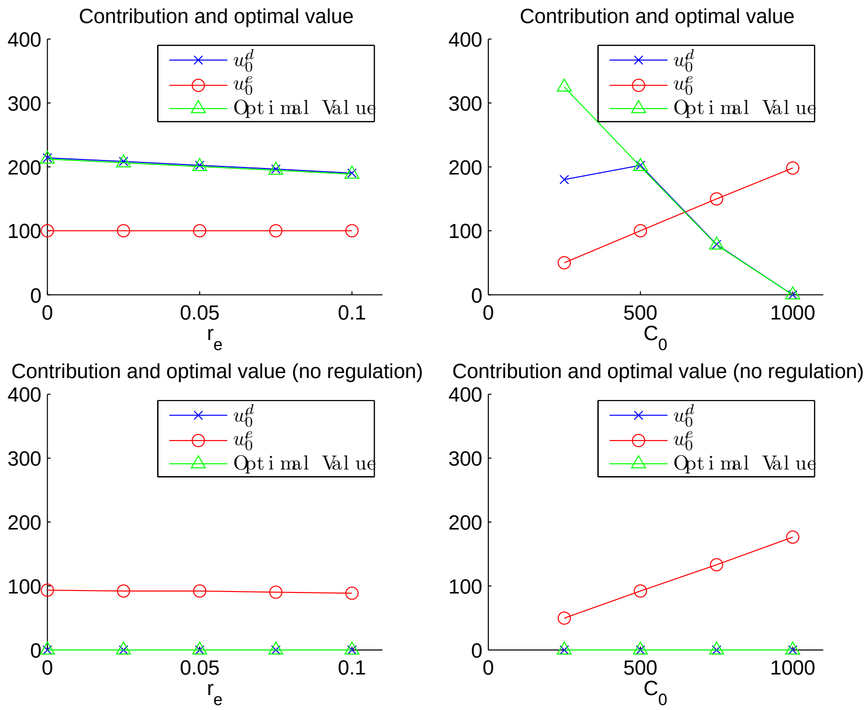

Figure 3: Contribution and optimal value for different scenarios

4.3 Stress testing an existing pension fund management strategy

Pension fund managers often want to calculate the projected contribution corresponding to a given funding level γ and a fixed portfolio holding policy, i.e. a given fixed $\{\boldsymbol{\theta}_k\}$. In this section, we show that our optimization model can be used to answer this question. Since the optimal solution of a robust optimization problem corresponds to the worst case realization of market performance, our model can be used to compute the worst case predicted contribution for a given portfolio holding strategy.

We again considered the planning horizon $\bar{N} = 4$ and $r_e = 0.05$. We assumed that the sponsor uses a fixed portfolio strategy, i.e. $\boldsymbol{\theta}_0 = \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = \boldsymbol{\theta}_3$. We computed $\boldsymbol{\theta}_0$ and $\{w_k\}_{k=0}^{\bar{N}}$ by solving

$$\begin{array}{rl} \min & \sum_{t=0}^{\bar{N}} \Big((1-T)B_{0,t} + \Big((1-T)P(\underline{Q}) + T\Big)B_{0,t+1}\Big) w_t^d \\ \text{subject to} & \boldsymbol{\theta}_0 = \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = \boldsymbol{\theta}_3, \\ & (B_{1,1}, \ldots, B_{1,M})(\theta_0(1), \ldots, \theta_0(M))^T = \rho q' \theta_0(M+1), \\ & (4),\ (18),\ (19),\ (7),\ (20),\ (10) \text{ and } (21). \end{array} \tag{23}$$

where the constraint $(B_{1,1}, \ldots, B_{1,M})\theta_t(1:M) = \rho q' \theta_0(M+1)$ sets the equity ratio of the initial portfolio $\boldsymbol{\theta}_0$ to $\frac{1}{(1+\rho)}$.

Table 7 and Figure 5 shows the worst-case payments as a function of the equity ratio $\frac{1}{(1+\rho)}$ with the probability of constraint satisfaction fixed at $p = 0.99$. The contribution w_0 increases with increasing equity ratio while the contributions (w_1, w_2, w_3) all decrease with the increase in the equity ratio. The total discounted payment, however, increases with the increase in the equity ratio.

$\mathbf{p}_0^T\boldsymbol{\psi}/L_0$	0.500	0.600	0.700	0.800	0.900	1.000	1.100	1.200	1.300	1.400
w_0^d	359.91	360.06	357.59	202.47	47.35	0.00	0.00	0.00	0.00	0.00
w_0^e	-0.12	-0.01	100.00	100.00	100.00	95.73	90.69	86.81	79.83	74.13
Equity Value	51.56	147.10	320.47	320.47	320.47	421.35	888.60	1269.09	1328.60	1369.31
Total Value	1306.25	1306.11	1443.45	1443.45	1443.45	1546.95	1697.04	1848.27	1996.42	2145.84
Equity Ratio	0.04	0.11	0.22	0.22	0.22	0.27	0.52	0.69	0.67	0.64
Optimal Value	3071.22	3103.92	354.28	200.59	46.91	0.00	0.00	0.00	0.00	0.00
$\sum_{t=0}^N B_{0,t}w_t$	3212.87	3246.08	676.88	521.76	366.61	306.30	295.65	286.03	266.48	247.40

Table 5: Optimal plan as a function of initial funding level ($r_e = 0.05, C_0 = 500$)

$\mathbf{p}_0^T\boldsymbol{\psi}/L_0$	0.500	0.600	0.700	0.800	0.900	1.000	1.100	1.200	1.300	1.400
w_0^d	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
w_0^e	100.00	100.00	99.00	92.13	83.81	83.03	81.37	80.39	76.94	72.33
Equity Value	0.00	0.00	7.94	120.87	423.59	870.42	1112.39	1278.73	1390.83	1516.34
Total Value	775.61	930.73	1084.86	1233.11	1379.91	1534.25	1687.72	1841.85	1993.53	2144.04
Equity Ratio	0.00	0.00	0.01	0.10	0.31	0.57	0.66	0.69	0.70	0.71
Optimal Value	287.83	134.23	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sum_{t=0}^N B_{0,t}w_t$	630.47	475.35	337.44	322.10	308.75	303.79	290.55	274.27	263.01	254.74

Table 6: Optimal plan as a function of initial funding level($r_e = 0.05$, $C_0 = 500$, and no regulation)

In Table 8 we display the worst-case payments as a function of the probability of constraint satisfaction with the equity ratio $\frac{1}{(1+\rho)}$ fixed at 0.4. As expected, the worst-case contribution decreases with a decrease in constraint satisfaction.

4.4 Conditional value-at-risk

In this section, we test the effect of linearizing the dynamics by evaluating the results in Section 4.3 using the Value-at-Risk and Conditional Value-at-Risk measures.

As in Section 4.3, we assume that the pension fund is committed to hold a given fixed portfolio $\boldsymbol{\theta}_0 = \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = \boldsymbol{\theta}_3$. We simulate the asset prices using the dynamics described by (1)-(3) (i.e. we do *not* linearize the dynamics) and compute the real (as opposed to the worst-case) payments $\bar{w}_t$ required to finance the portfolio strategy. From the constraints (4), (5), (6), (7) and (8), it follows that

$$\bar{w}_t = \begin{cases} \max(l_t + \mathbf{p}_t^T\boldsymbol{\theta}_t - \alpha_{t-1}(\mathbf{p}_t^T\mathbf{D}\boldsymbol{\theta}_{t-1} + \mathbf{d}^T\boldsymbol{\theta}_{t-1}), \gamma L_t - \mathbf{p}_t^T\boldsymbol{\theta}_t, 0) & \text{if } 1 \le t < \bar{N}, \\ \max(l_t + \beta L_t - \alpha_{t-1}(\mathbf{p}_t^T\mathbf{D}\boldsymbol{\theta}_{t-1} + \mathbf{d}^T\boldsymbol{\theta}_{t-1}), 0) & \text{if } t = \bar{N}, \end{cases} \tag{24}$$

Equity ratio	w_0	w_1	w_2	w_3	$\sum_{t=0}^{\bar{N}} B_{0,t}w_t$
0.2	239.90	125.00	155.60	180.41	683.69
0.4	283.96	120.04	143.91	163.44	697.36
0.6	348.05	114.88	131.76	145.81	729.92

Table 7: Worst-case contribution as a function of equity ratio

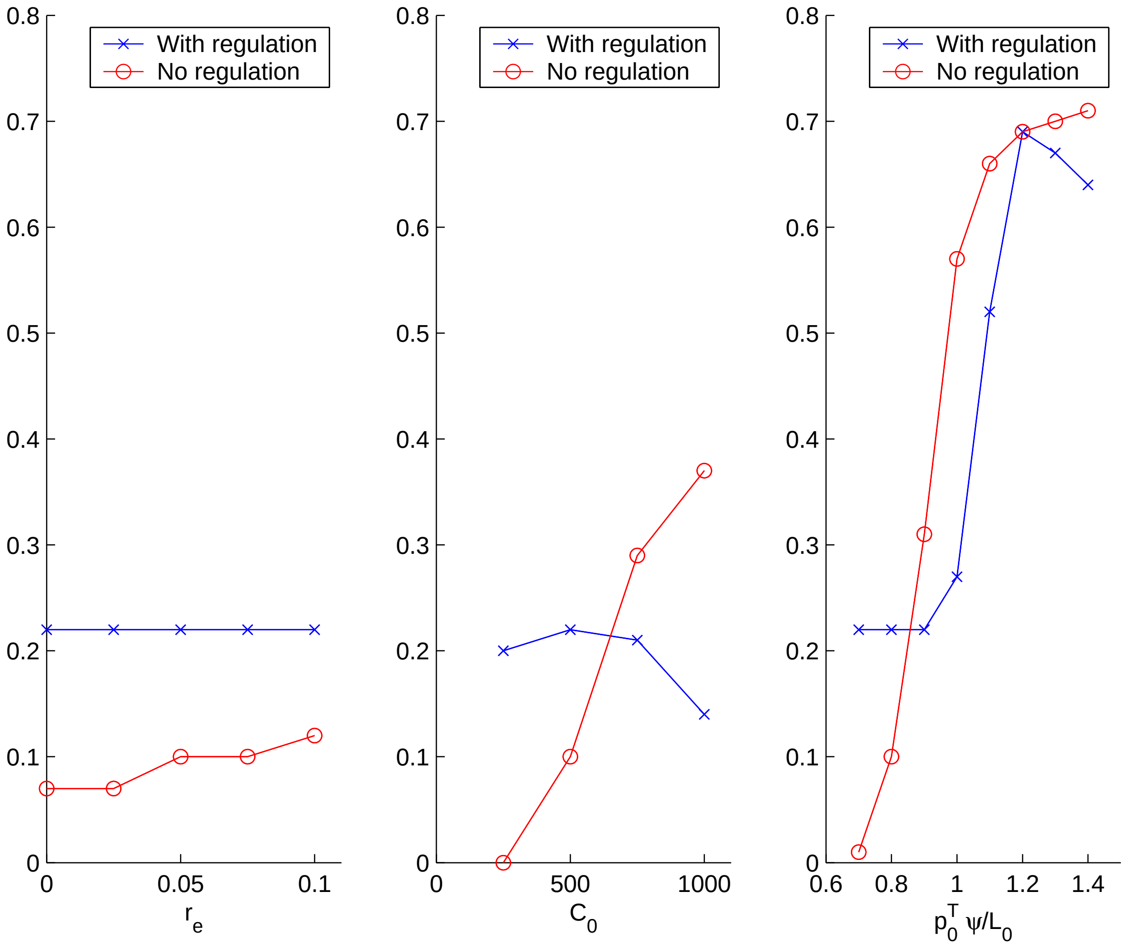

Figure 4: Equity ratio for different scenarios

where

$$\alpha_t = \max\left(\frac{\gamma L_t}{\mathbf{p}_t^T \boldsymbol{\theta}_t}, 1\right). \tag{25}$$

The variable α_t keeps track of whether or not the payment $\bar{w}_t$ is needed to maintain the regulation requirement $\frac{\gamma L_t}{\mathbf{p}_t^T \boldsymbol{\theta}_t} \leq 1$, and the value of the portfolio in the next period will increase or remain unchanged accordingly. Note that in our numerical experiments, θ_t is fixed over time.

We generated $K = 100,000$ independent sample paths and set the shortfall probability

$$\bar{p} = \frac{\sum_{k=1}^{K} \max_{0 \leq t \leq 3} \mathbf{1}(w_t < \bar{w}_t^{(k)})}{K},$$

where $\{\bar{w}_t^{(k)}\}$ denotes the real payments on the k-th simulation run and $\mathbf{1}(\cdot)$ is the indicator function that takes the value 1 when the argument is true and 0 otherwise. Thus, $\bar{p}$ is the empirical probability that the real payment $\bar{w}_t$ is larger than the worst case payment w_t. The expected net shortfall $\bar{W}$ was defined as follows.

$$\bar{W} = \frac{\sum_{k=0}^{K} \sum_{t=0}^{\bar{N}} B_{0,t}(\bar{w}_t^{(k)} - w_t)^+}{\sum_{k=0}^{K} \mathbf{1}(\sum_{t=0}^{\bar{N}} B_{0,t}(\bar{w}_t - w_t)^+ > 0)}, \tag{26}$$

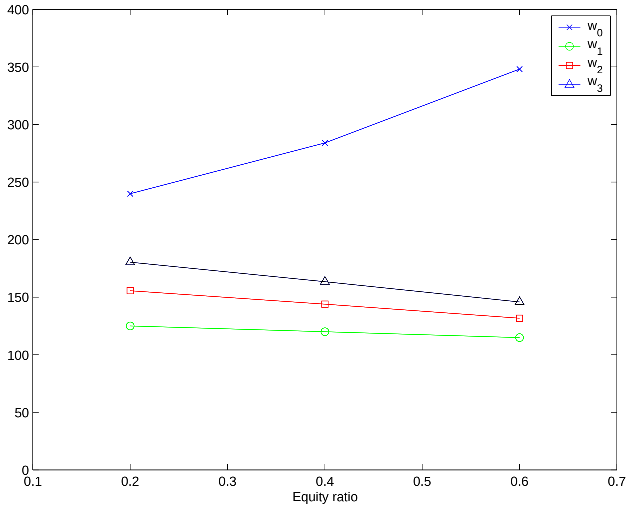

Figure 5: Worst-case contribution as a function of equity ratio

Probability	w_0	w_1	w_2	w_3	$\sum_{t=0}^{\bar{N}} B_{0,t} w_t$
0.99	283.96	120.04	143.91	163.44	697.36
0.95	244.27	111.20	130.97	147.34	622.91
0.90	230.03	107.11	125.01	139.95	592.79

Table 8: Worst-case contribution as a function of time

i.e. $\bar{W}$ is the expected shortfall conditioned on their being a shortfall. We define the Value-at-Risk (VaR_p) at probability p of the discounted total real payment $\sum_{i=0}^{\bar{N}} B_{0,t}\bar{w}_t$ as

$$\text{VaR}_p = \sup_{x \geq 0} \left\{ \sum_k \mathbf{1}\Big(\sum_{i=0}^{\bar{N}} B_{0,t}\bar{w}_t^{(k)} \geq x \Big) \geq (1-p)K \right\}$$

and Conditional Value-at-Risk (CVaR_p) of the discounted total real payment as

$$\text{CVaR}_p = \frac{1}{(1-p)} \left(\sum_{k=1}^{K} \Big(\sum_{i=0}^{\bar{N}} B_{0,t}\bar{w}_t^{(k)} \Big) \mathbf{1}\Big(\sum_{i=0}^{\bar{N}} B_{0,t}\bar{w}_t^{(k)} \geq \text{VaR}_p \Big) \right).$$

Table 9 plots the shortfall probability $\bar{p}$, the expected shortfall $\bar{W}$, the VaR and CVaR as a function of the probability p that is used to set the bounds ϵ_j in the robust problem. From the numerical results, we can conclude that

(a) The linearized robust problem (23) does produce a *conservative* solution for the true non-linear problem (note that this is not guaranteed). In all cases, the empirical shortfall probability is at least an order of magnitude lower than that guaranteed by the robust problem. This result confirms our initial hypothesis that linearizing the dynamics should not result in a significant deterioration in performance.

(b) For a fixed p, let $\tilde{p}$ denote the probability such that the corresponding shortfall probability $\bar{\tilde{p}} \approx 1-p$. For example, for $p = 0.98$, $\tilde{p} = 0.85$ since the corresponding shortfall probability

p	$\bar{p}$	$\bar{W}$	$\sum_{t=0}^{\bar{N}} B_{0,t} w_t$	VaR	CVaR
0.99	0.0014	5.98	697.36	537.40	546.96
0.98	0.0027	5.07	664.48	511.58	521.73
0.97	0.0039	5.41	644.63	496.18	506.83
0.96	0.0050	5.34	632.27	487.12	498.22
0.95	0.0063	5.43	622.91	479.91	491.22
0.90	0.0126	5.64	592.79	456.23	469.00
0.85	0.0182	5.99	574.24	441.90	455.63
0.80	0.0235	6.08	560.48	430.38	445.12

Table 9: Simulation results

$\bar{\bar{p}} = 0.0182 \approx 1-p = 0.02$. Another such pair is $(p, \tilde{p}) = (0.99, .090)$. Then total discounted worst case payment corresponding to $\tilde{p}$ is approximately equal to the CVaR_p – note that this is in spite of the fact that the robust problem does *not* minimize the total discounted payment.

4.5 Computational efficiency

All numerical computations reported in this work were conducted using Matlab 6.5 and MOSEK 4.0 (Andersen and Andersen, 2006). We used a Windows/32-X86 platform with Intel-PM. A typical portfolio problem had less than 100 constraints and 100 variables and it took no longer than a second for MOSEK to solve the portfolio problem.

5 Concluding Remarks

In this paper we develop a robust-optimization based model for pension fund management that minimizes the worst-case pension contributions of the sponsoring firm. This allows us to provide worst-case guarantees. The model is able to account for some aspects of the corporate structure of the firm, in particular its cost of debt. The optimal pension plan in our model is computed by solving an SOCP and is, therefore, very efficient both in theory and in practice. In Section 4 we show that the model is very versatile in that it allows us to compute both the optimal plan and also stress test any existing pension plans.

Our model does have some limitations that need to be addressed. We assume that the liabilities are fixed and known. While this might be the case with *frozen* pension funds, in most funds the liability is subject to actuarial uncertainty. Our model does not account for this uncertainty. It is not clear how one could hedge this uncertainty as it is likely to be uncorrelated with the market. While we do take the cost of debt into consideration, we do not take the entire firm strategy into account. This gap does manifest itself. For example, when $u = 1$ the optimal pension plan may use up the entire operating revenue. This happens because we do not account for other operating requirements of the firm.

While the model does have limitations, our model has many attractive features and provides reliable solutions in situations where sampling-based methods become intractable.

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References

(2006). H.r. 4 — pension protection art 2006. http://rpc.senate.gov/_files/L53HR4Pensions080206DK.pdf.

(2006). The other pension crisis. *The Wall Street Journal*.

Alizadeh, F. and Goldfarb, D. (2003). Second-order cone programming. *Mathematical Programming*, 95:3–51.

Andersen, E. D. and Andersen, K. D. (2006). *The MOSEK optimization toolbox for MATLAB manual Version 4.0*. http://www.mosek.com/products/4_0/tools/help/index.html.

Barrett, W. B., Gosnell, T., and Heuson, A. (1995). Yield curve shifts and the selection of immunization strategies. *Journal of Fixed Income*, pages 53–64.

Ben-Tal, A., Margalit, T., and Nemirovski, A. (2000). Robust modeling of multi-stage portfolio problems. *High performance optimization*, pages 303–328.

Ben-Tal, A. and Nemirovski, A. (2001). *Lectures on modern convex optimization: analysis, algorithms, and engineering applications*. Society for Industrial & Applied Math.

Consigli, G. and Dempster, M. (1998). Dynamic stochastic programming for asset-liability management. *Annals of Operations Research*, 81:131–161.

Damodaran, A. (2004). *Applied Corporate Finance: a user's manual*. Wiley, 2nd edition.

Drijver, S., Haneveld, W., and Vlerk, M. (2000). Asset liability management modeling using multi-stage mixed-integer stochastic programming. Technical report, University of Groningen.

Fabozzi, F., Focardi, S., and Jonas, C. (2004). Can modeling help deal with the pension funding crisis. The Intertek Group.

Fabozzi, F., Focardi, S., and Jonas, C. (2005a). Market expericence with modeling for defined-benefit pension funds: evidence from four countries. *Journal of Pension Economics and Finance*, pages 313–327.

Fabozzi, F., Martellini, L., and Priaulet, P. (2005b). "predictability in the shape of the term structure of interest rates". *Journal of Fixed Income*, pages 40–53.

Fabozzi, F., Martellini, L., and Priaulet, P. (2005c). Preditability in the shape of the term structure of interest rates. *Journal of Fixed Income*.

Gladwell, M. (2006). Risk pool. *New Yorker*.

Goldfarb, D. and Iyengar, G. (2003). Robust convex quadratically constrained programs. *Mathematical Programming Ser B.*, 97(3):495–515.

Klaassen, P. (1998). Financial asset-pricing theory and stochastic programming models for asset/liability management: a synthesis. *Management Science*, 44(1):31–48.

Kloeden, P. and Platen, E. (1999). *Numerical solution of stochastic differential equations*. Springer.

Mulvey, J. M., Gould, G., and Morgan, C. (2000). An asset and liability management system for towers perrin-tillinghast. *Interfaces*, 30(1):96–114.

Myers, S. and Majluf, N. (1984). Corporate financing and investment decisions when firms have information that investors do not have. *Journal of Financial Economics*, 13(2):155–295.

Nelson, C. and Siegel, A. (1987). Parsimonious modeling of yield curves. *Journal of Business*, 60(4):473–489.

Sodhi, M. (2005). LP modeling for asset-liability management: a survey of choices and simplifications. *Operations Research*, 53(2):181–196.