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Approximate formulations for 0-1 knapsack sets

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Abstract

A classical theorem in Combinatorial Optimization proves the existence of *fully polynomial-time approximation schemes* for the knapsack problem [2], [3]. In a recent paper [4], Van Vyve and Wolsey ask whether for each $0 < \epsilon \le 1$ there exists an extended formulation for the knapsack problem, of size polynomial in the number of variables and/or ϵ^{-1}, whose value is at most $(1+\epsilon)$ times the value of the integer program. In this note we partially answer this question in the affirmative, using techniques similar to those in [1].

1 Introduction

Consider the feasible set for a $0-1$ knapsack problem,

$$\sum_{j=1}^{n} a_j x_j \ \le\ a_0, \quad x \in \{0\,,1\}^n, \tag{1}$$

where $a_j \ge 0$ for $0 \le j \le n$. Here we prove the following result:

Theorem 1.1 *Let $0 < \epsilon \le 1$. There exists an extended formulation*

$$Ax + By + Cz \ \le\ b, \tag{2}$$

with $O\left(\epsilon^{-1} n^{1+\lceil 1/\epsilon \rceil}\right)$ variables and $O\left(\epsilon^{-1} n^{2+\lceil 1/\epsilon \rceil}\right)$ constraints such that

$$\left\{ x \in \{0,1\}^n \,:\, \sum_{j=1}^{n} a_j x_j \le a_0 \right\} \ \subseteq\ \{ x \in R^n \,:\, \exists (y,z)\ s.t.\ Ax + By + Cz \le b \}, \tag{3}$$

and for any $w \in R^n_+$,

$$\max \left\{ w^T x \,:\, \sum_{j=1}^{n} a_j x_j \ \le\ a_0,\ x \in \{0\,,1\}^n \right\} \ \ge\ (1-\epsilon) \max \left\{ w^T x \,:\, \exists (y,z)\ s.t.\ Ax + By + Cz \le b \right\}.$$

2 The construction

Let $H = \left\lceil \frac{1}{\epsilon} \right\rceil$. We assume $n \ge H$. The variables y, z in the theorem are constructed as follows.

(a) For each integer $0 \le h < H$, and each subset $S \subseteq \{1, 2, \ldots, n\}$ with $|S| = h$, we have variables y^S_j, for $0 \le j \le n$, as well as the constraints:

$$y^S_j \ \ge 0,\ \ 0 \le j \le n, \tag{4}$$

$$y^S_j \ -\ y^S_0 \ = 0,\ \ \forall\, j \in S, \tag{5}$$

$$y^S_j \ = 0,\ \ \forall\, j \notin S \cup \{0\}, \tag{6}$$

$$\sum_{j=1}^{n} a_j y^S_j \ -\ a_0\, y^S_0 \ \le\ 0. \tag{7}$$

(b) For each each subset $S \subseteq \{1, 2, \ldots, n\}$ with $|S| = H$, we have variables z_j^S, for $0 \leq j \leq n$, as well as the constraints:

$$z_j^S \geq 0, \;\; 0 \leq j \leq n, \tag{8}$$
$$z_j^S \leq z_0^S, \;\; 1 \leq j \leq n, \tag{9}$$
$$z_j^S - z_0^S = 0, \;\; \forall j \in S, \tag{10}$$
$$z_j^S = 0, \;\; \text{if}\, j \notin S \cup \{0\} \;\text{ and }\; a_j > \min_{i \in S}\{a_i\}, \tag{11}$$
$$\sum_{j=1}^{n} a_j z_j^S - a_0 \, z_0^S \leq 0. \tag{12}$$

(c) In addition, we have the constraints:

$$\sum_S y_j^S + \sum_S z_j^S - x_j = 0, \;\; 1 \leq j \leq n, \tag{13}$$
$$\sum_S y_0^S + \sum_S z_0^S = 1. \tag{14}$$

where these sums are understood to run over appropriate indices as defined in (a) and (b).

Lemma 2.1 *Constraints (4)-(14) define a valid relaxation for (1), i.e. the projection of the feasible set for (4)-(14) to the space of the x variables contains the feasible set for (1).*

Proof. Consider a 0-1 vector $\hat{x}$ satisfying (1). Let $\hat{S} = \{1 \leq j \leq n : \hat{x}_j = 1\}$.

Suppose first that $|\hat{S}| < H$. Then we define $y_j^{\hat{S}} = \hat{x}_j$ for $1 \leq j \leq n$, and $y_0^{\hat{S}} = 1$; and set $y_j^S = 0$ for all other sets S and all j, and all $z_j^S = 0$. Note that this argument is correct even when $\hat{S} = \emptyset$.

Suppose now that $|\hat{S}| > H$. Let $\bar{S} \subset \hat{S}$ consist of the H indices $j \in \hat{S}$ with largest a_j (ties arbitrarily broken). Then we set $z_j^{\bar{S}} = 1$ for all $j \in \hat{S}$, $z_0^{\bar{S}} = 1$, and set $z_j^S = 0$ for all other combinations of S and j; and all $y_j^S = 0$. ∎

Write $W^* = \max\left\{w^T x \,:\, \sum_{j=1}^n a_j x_j \leq a_0, \; x \in \{0, 1\}^n\right\}$.

Lemma 2.2 *Suppose $(\hat{x}, \hat{y}, \hat{z})$ satisfy (4)-(14). Let $w \in R_+^n$. Then*

(i) For any set S included in case (a) of the construction,

$$W^* \hat{y}_0^S \geq \sum_{j=1}^{n} w_j \hat{y}_j^S. \tag{15}$$

(ii) For any pair k, S included in case (b) of the construction,

$$W^* \hat{z}_0^S \geq (1 - \epsilon) \sum_{j=1}^{n} w_j \hat{z}_j^S. \tag{16}$$

Proof. (i) If $\hat{y}_0^S = 0$ the result is clear, and if $\hat{y}_0^S > 0$ then the $0-1$ vector with entries $\hat{y}_j^S / \hat{y}_0^S$ $(1 \leq j \leq n)$ satisfies (1) from which the result follows.

(ii) As in (i) assume that $\hat{z}_0^S > 0$, and define $\bar{x}_j = \hat{z}_j^S / \hat{z}_0^S$ for $1 \leq j \leq n$. By construction in case (b), we have that $\bar{x}$ is a feasible solution to the linear program:

$$\tilde{W} \doteq \max \sum_{j=1}^{n} w_j x_j \tag{17}$$

Subject to: (18)

$$0 \le x_j \le 1, \;\; 1 \le j \le n, \tag{19}$$

$$x_j = 1, \;\; \forall\, j \in S, \tag{20}$$

$$x_j = 0, \;\; \text{if}\, j \notin S \;\text{ and }\; a_j > \min_{i \in S}\{a_i\}, \tag{21}$$

$$\sum_{j=1}^{n} a_j x_j \le a_0. \tag{22}$$

Thus, in order to conclude with case (ii) it suffices to prove that $W^* \ge (1-\epsilon)\tilde{W}$. To this end, let $\tilde{x}$ be an extreme point optimal solution to the LP (17)-(22). We assume $\tilde{x}$ is not integral for otherwise the result is clear.

Clearly, there exists exactly one index p such that $0 < \tilde{x}_p < 1$.

Let $i = \text{argmin}_{j \in S}\{w_j\}$, and suppose that $w_i < w_p$. Then we increase $\tilde{x}_p$ by $1 - \tilde{x}_p$, decrease $\tilde{x}_i$ by $1 - \tilde{x}_p$, and reset $S \leftarrow S - \{i\} \cup \{p\}$. By (21), we have $a_i \ge a_p$. Thus, after the change, the vector $\tilde{x}$ still satisfies (22), as well as (19). Moreover, the objective value of $\tilde{x}$ has increased.

Thus (whether the change was performed or not), we have:

(C.1) $0 < \tilde{x}_q < 1$ for one entry q,

(C.2) There is a set S with $|S| = H$ such that $\tilde{x}_i = 1$ for all $i \in S$. and if an index q as in (C.1) exists, then $w_q \le \min_{i \in S}\{w_i\}$.

(C.3) $\tilde{x}$ satisfies (22),

(C.4) $\sum_j w_j \tilde{x}_j \ge \tilde{W}$.

Consider the $0-1$ vector $\tilde{x}$ defined by $\tilde{x}_j = \lfloor \tilde{x}_j \rfloor$ for $1 \le j \le n$. By (C.3) this vector is feasible for the knapsack constraint (1). Furthermore, by (C.1) and (C.2), we have that

$$\frac{\sum_j w_j \tilde{x}_j - \sum_j w_j \tilde{x}_j}{\sum_j w_j \tilde{x}_j} \le \frac{1}{H} \le \epsilon, \tag{23}$$

and therefore

$$(1-\epsilon) \sum_j w_j \tilde{x}_j \le \sum_j w_j \tilde{x}_j \le W^*, \tag{24}$$

as desired. ■

Lemma (2.2), together with constraints (13) and (14) of our system, complete the proof of Theorem 1.1.

References

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