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Bidding strategically with budget-constraints in sequential auctions

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Abstract

We examine models of sequential auctions with budget-constrained bidders. These types of auctions are of particular importance in e-commerce. Our models are simple, but rich enough to posess interesting features such as bid-jamming, which is also found in internet search engine auctions. We apply adversarial analysis to find fixed-point bidding strategies which are distribution-free and randomized strategies for when the wealth of the bidder is low relative to her competitors.

1 Introduction

Multi-round auctions with budget-constrained bidders appear in a variety of contexts. Search engines such as Google, Yahoo, and MSN now auction off search terms to potential advertisers. The potential advertisers place their bids on each search term of interest, as well as specifying a daily budget. Each search on this term displays an advertisement that is linked to the advertiser's website, and the advertiser pays the search engine every time the link is activated. When an advertiser's budget is reached, the search engine stops displaying their ad. This kind of advertising is extremely popular – the combined revenue of Yahoo and Google in 2005 was estimated at over 4.5 billion dollars [10]. Another example of a multiround auction with budget-constrained bidders is the FCC spectrum auction (see Cramton [7]) where companies bid and purchased rights to bandwidth ranges for wireless communication services.

We consider a multi-round auction where each of the bidders has a fixed budget constraint. We examine this problem from the perspective of the bidder, rather than the auctioneer, and seek to develop strategies for the bidder with respect to a given auction design. The problem of determining the best bid for symmetric bidders with budgets and values drawn from independent distributions in one round is well known (e.g., see Krishna [9]). However, the situation when there are asymmetric bidders with budget constraints is both non-trivial and not well understood. For example, both Pitchik [13] and Che and Gale [5, 6] show how the revenue equivalence theorem (see [9]) no longer holds in this case. Pitchik [13] shows how the order of sale in a two-object sequential auction can change the outcome. Che and Gale [5, 6] and Benoit and Krishna [2] consider the problem of designing auctions with budget-constrained bidders.

To the best of our knowledge, there has been little prior work on bidding strategies in multi-round budget-constrained auctions. Oren and Rothkopf [12] consider a multi-period model without budget constraints where each bidder is assumed to have a function that describes their beliefs about the bidding strategies of their competitors. Consequently, the problem of computing an equilibrium can be reformulated as a dynamic program. We know of no other results related to how bidder's should choose bids in the multi-round budget-constrained setting.

A further difficulty of the auction situation we model, is that bidders do not place bids simultaneously, as is assumed in previous models ([9]). Rather, bidders place bids in a sequential fashion, and the auctioneer must resolve bids as they occur. To account for this difference, we model the auction as a *Stackleberg game* (see von Stackleberg [14]). Such models have been used in military applications, specifically the network interdiction problem (see Brown et al [3]), for which some complexity results are known (see Brown et al [4]).

A significant difficulty of this problem is that there are two distinct strategies for most common mechanisms (e.g., first-price and second-price): the bidders can choose to win an auction outright, or decide to bid so as to strip their competitors of budget. In the latter strategy, for a second-price auction, the bidder will seek to bid just below their competitor to force the competitor to pay as much as possible. In keyword bidding problems, this is called Bid Jamming, and there are even software tools which allow bidders to automatically jam their competitors (e.g., see [1]). Our goal was to develop models which would retain this feature of the problem, but would allow us to generate non-trivial strategies.

Our setting is simple: two bidders, one object with a common value, and two rounds of bidding. The budget of one of the bidders is fixed, and the other is distributed according to an arbitrary distribution. Despite the model's simplicity, this setting still retains the complexity of the strategy of jamming versus winning in the first round. Moreover, even simple models with budget budget constraints have proven to be quite difficult.

To develop bidding strategies in this setting, we use the concept of adversarial analysis (see, e.g., [11]) to compute provably competitive strategies. This technique assumes that one of the bidders, the adversary, has more information than the other bidder, called the player. The idea of asymmetric information in a economic setting is not new, e.g., see Kagel and Levin [8]. However, we know of no other previous results where this model was used to derive bidder strategies.

The objective is to compute the minimax optimal strategy and the corresponding payoff for both the player and the adversary. Such an adversarial analysis is conservative, i.e., worst-case, in nature. Our results focus on developing effective player strategies against this strong adversary. The benefit of this technique, in addition to allowing tractable analysis, is that the adversary will be able to jam the player very effectively, which is precisely the behavior we wish to try and strategize against.

Our results are as follows:

(a) We develop a fixed strategy which is weakly dominating as long as the adversary's budget is bounded by 1.5 times the player's budget. This strategy is independent of distribution.

(b) We describe a simple randomized strategy which dominates the fixed strategy, and describe a natural example where the randomized strategy performs significantly better than the fixed strategy by reducing the ability of the adversary to jam.

2 Notation

The following details are shared by both of our models.

(a) 2 bidders;

Bidder 1: The player; and

Bidder 2: The adversary; (who is assumed to have more information. see below.)

(b) 2 rounds of bidding;

(c) 2 identical objects to be sold each round;

(d) Second price bidding resolution at the end of each round;

(e) Ties are resolved in favor of the player.

We borrow the following conventions from economics: Suppose the n-dimensional vector

$$\mathbf{x}^\top = (x_1, x_2, \ldots, x_n),$$

is given. Then, as usual, x_i will denote the ith vector. We let $\mathbf{x}_{-i}$ denote the $n-1$ vector identical to $\mathbf{x}$ but missing the ith component. Further, we let

$$(y, \mathbf{x}_{-i}^\top) = (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n),$$

which is the vector $\mathbf{x}$ with the ith component replaced by the value y. For the case of $n = 2$, we will use x_{-i} to denote the $[2-(3-i)]$th component. Thus, x_{-i} is the "other" component with respect to x_i.

For a boolean statement, $\mathcal{S}$, we will let $\mathbf{1}$ denote the indicator function for this statement. Thus,

$$\mathbf{1}(\mathcal{S}) = \begin{cases} 1 & \mathcal{S} \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

We use the following notation:

$$\begin{aligned}
w_i, W_i \quad & = \text{the budget (or wealth) of bidder } i, \\
& W_1 = 1 \\
& W_2 \sim F, \text{ cumulative distribution function on } [0, \omega], \\
v \quad & = \text{the common value of the two objects}, \\
x_i^t \quad & = \text{the bid of bidder } i \text{ in round } t, \\
u_i^t \quad & = \text{the profit function to bidder } i \text{ in round } t, \\
& = \begin{cases} v - x_{-i}^t & \text{if } x_i^t \geq x_{-i}^t \\ 0 & \text{otherwise.} \end{cases} \\
u_i \quad & = \text{the total profit function to bidder } i \\
& = u_i^1 + u_i^2 \\
r_i \quad & = \text{the remaining budget function of bidder } i \text{ after round 1 is complete} \\
& = w_i - x_i^1
\end{aligned}$$

When we discuss a sample path realization of the adversary's budget we will use the lowercase w_2. The reason for this distinction is that the adversary knows his budget before choosing a bid, whereas the player knows only the distribution of the adversary's budget before choosing a bid or strategy.

All functions listed are real-valued on the spaces $[0, w_i], i = 1, 2$ for round 1 and $[0, r_i], i = 1, 2$ for round 2. The following lemma shows that the overall profit function is only dependent on the first round bids.

We also assume that each bidder is strongly budget-constrained:

$$x_i^1 \leq w_i, x_i^2 \leq r_i.$$

Lemma 1 (Krishna [9]) *Suppose that round 1 has been resolved. Then a weakly dominant strategy in round 2 for each bidder i is:*

$$x_i^2 := \min\{v, r_i\}. \tag{1}$$

Proof:

After round 1 has been resolved, our model has become a single period model with new budgets,

$$r_i = b_i - c_i^1.$$

The second-round profit for the player is:

$$u_1^2(x_1^2, x_2^2) = \begin{cases} v - x_2^2 & x_1^2 \geq x_2^2. \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Suppose the player bids $x_1^2 < \min\{v, r_1\}$. Then, if $x_2^2 \in (x_1^2, \min\{v, r_1\})$, the player wins nothing, but could have won $v - x_2^2 > 0$ by bidding $\min\{v, r_1\}$. So, instead, suppose the player bids $x_1^2 > \min\{v, r_1\}$. Note that $x_1^2 \leq r_1$, whence $v = \min\{v, r_1\}$ and $v < x_1^2 \leq r_1$. Then, if $x_2^2 \in (v, x_1^2)$ the player has a negative profit of $v - x_2^2 < 0$, and would have been better off bidding v.

The second-round profit for the adversary is:

$$u_2^2(x_1^2, x_2^2) = \begin{cases} v - x_1^2 & x_1^2 < x_2^2. \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Despite the inequality being strict, the logic from the player's case works here as well. Bidding less than $\min\{v, r_2\}$ is dominated by (1) when $x_1^2 \in (x_2^2, \min\{v, r_2\})$ and bidding greater than v is dominated by (1) when $x_1^2 \in (v, x_2^2)$. ■

Note that this argument is true whether or not the bidder is aware of the other bidder's bid or not. Also, Lemma 1 indicates that the only decision the Adversary and Player need to make is their bid in round 1. Thus, a strategy is completely determined by the bids in round 1. Therefore, we adopt redefine our notation and let:

$$x_i = \text{the round one bid of player } i, i = 1, 2.. \tag{4}$$

Note that we still use superscripts for the second round bids, so x_i^2 denotes the second round bid for bidder i as determined by Lemma 1. We can also write equations for the utility for each player based on the first round bids.

The player's profit is:

$$\begin{aligned} u_1(x_1, x_2) \quad := \quad & \mathbf{1}\{x_1 \geq x_2\}\big(v - x_2 + \mathbf{1}\{1 - x_2 \geq W_2\}(v - W_2)\big) \\ & + \mathbf{1}\{x_1 < x_2\}\mathbf{1}\{W_2 - x_1 \leq 1\}(v - W_2 + x_1) \end{aligned} \tag{5}$$

The adversary's profit is:

$$\begin{aligned} u_2(x_1, x_2) \quad := \quad & \mathbf{1}\{x_1 < x_2\}\big(v - x_1 + \mathbf{1}\{W_2 - x_1 \geq 1\}(v - 1)\big) \\ & + \mathbf{1}\{x_1 \geq x_2\}\mathbf{1}\{1 - x_2 \leq W_2\}(v - 1 + x_2) \end{aligned} \tag{6}$$

3 The first model

The first model can be described from the perspective of the sequence of bidding or the perspective of information availability. From the perspective of the sequence of bidding, the model is a variant of a Stackleberg game. In particular, the bidding and resolution works as follows in each round:

1. The player bids a fixed amount x_1.

	Round	Information known prior to bidding	Hidden Information	Action
Player	1	$v, W_1 = 1, F$	x_2, W_2	fixed bid
	2	r_1, x_2	x_1^2, x_2^2, r_2	
Adversary	1	$v, W_1 = 1, W_2, x_1$		random bid
	2	v, r_1, r_2, x_1^2		

Figure 1: Model 1

2. With the knowledge of the player's bid, the adversary bids x_2 possibly from specifying a random distribution from which to choose his bid.

3. The higher bidder wins paying the lower bid (second-price).

Recall that, by Lemma 1, the second round bid is determined by the first round bids and resolution. From the perspective of the availability of information, the model can be described by Figure 1: The information revealed to each bidder is not "forgotten," so in round 2, each bidder knows everything from round 1.

Theorem 1 *Given that the player bids x_1 in round 1,a weakly dominant strategy for the adversary is to bid*

$$x_2 := \left\{ \begin{array}{ll} w_2 & w_2 \leq x_1; \\ \hline x \in (x_1, w_2] & \begin{array}{l} x_1 < w_2 \leq 1 + x_1, x_1 \leq 1/2, \\ w_2 > 1 + x_1, x_1 \leq v/2 \text{ or } \ v \leq 1 - x_1; \ \textit{and} \end{array} \\ \hline x_1 & \begin{array}{l} x_1 < w_2 \leq 1 + x_1, x_1 > 1/2, \\ w_2 > 1 + x_1, x_1 > v/2 \text{ and } \ v > 1 - x_1. \end{array} \end{array} \right. \tag{7}$$

Proof: We first note that there is never any incentive for the player to choose $x_1 > v$. If she did so, then the adversary would bid x_1 as well, causing the player to incur negative profit, and lose budget for the second round. By bidding a smaller amount, the player would be able to decrease the negative profit incurred and increase the chance of winning in the second round. We therefore assume that

$$x_1 \leq \min\{1, v\}. \tag{8}$$

We will prove the theorem using case analysis.

(a) $w_2 \leq x_1$
By (8), $w_2 \leq x_1 \leq v$. Consider the profit function:

$$\begin{aligned} u_2(x_1, x_2) &= \mathbf{1}\{1 - x_2 < w_2\}\mathbf{1}\{1 - x_2 < v\}(v - (1 - x_2)), \\ &= \begin{cases} v + x_2 - 1, & \min\{v, w_2\} > 1 - x_2, \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} v + x_2 - 1, & w_2 + x_2 > 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{9}$$

This is clearly maximized when $x_2 = w_2$. In this case, the player's profit is:

$$u_1(x_1, w_2) = (1 + \mathbf{1}\{w_2 \leq 1/2\})(v - w_2). \tag{10}$$

(b) $x_1 < w_2 \leq 1 + x_1, x_1 \leq 1/2$, In this case, the profit function is:

$$\begin{aligned} u_2(x_1, x_2) &= \begin{cases} v - x_1 + \mathbf{1}\{w_2 - x_1 > 1\}(v - \min\{v, 1\}), & x_2 > x_1, \\ \mathbf{1}\{w_2 > 1 - x_2\}\mathbf{1}\{v > 1 - x_2\}(v - 1 + x_2), & x_2 \leq x_1, \end{cases} \\ &= \begin{cases} v - x_1, & x_2 > x_1, \\ \mathbf{1}\{w_2 + x_2 > 1\}\mathbf{1}\{v > 1 - x_2\}(v - 1 + x_2) & x_2 \leq x_1. \end{cases} \end{aligned}$$

But, since $x_1 \leq 1/2, -x_1 \geq x_1 - 1$, and, if $x_2 \leq x_1$,

$$v - x_1 \geq v - 1 + x_1 \geq v - 1 + x_2.$$

This is an upper bound on the profit obtained by the adversary bids $x_2 \leq x_1$. Thus, the adversary's best strategy is to bid some value greater than x_1 and he receives a profit of

$$u_2(x_1, x_2) = v - x_1. \tag{11}$$

In this case, the player receives a profit of

$$u_1(x_1, x_2) = v - \min\{v, w_2 - x_1\}. \tag{12}$$

(c) $x_1 < w_2 \leq 1 + x_1, x_1 > 1/2$ The profit function in this case is:

$$\begin{aligned} u_2(x_1, x_2) &= \begin{cases} v - x_1 + \mathbf{1}\{w_2 - x_1 > 1\}(v - \min\{v, 1\}), & x_2 > x_1, \\ \mathbf{1}\{w_2 > 1 - x_2\}\mathbf{1}\{v \geq 1 - x_2\}(v - 1 + x_2), & x_2 \leq x_1, \end{cases} \\ &= \begin{cases} v - x_1, & x_2 > x_1, \\ (v - 1 + x_2), & x_2 \leq x_1. \end{cases} \end{aligned}$$

In the second case, the optimal choice for the adversary is to let $x_2 = x_1$. With this choice, $w_2 \geq x_2 = x_1 > 1/2$ and $w_2 + x_2 > 1$, and $v \geq x_1 = x_2 > 1/2 \geq 1 - x_2$, as $x_1 \leq 1$. Moreover, $x_2 - 1 > x_1 - 1 > -x_1$, whence

$$v - 1 + x_2 > v - x_1,$$

and the adversary should bid $x_2 = x_1$. His profit is

$$u_2(x_1, x_1) = v - \min\{v, 1 - x_1\}. \tag{13}$$

The player receives a profit of

$$u_1(x_1, x_1) = v - x_1. \tag{14}$$

(d) $w_2 > 1 + x_1, x_1 \leq v/2$. The profit function in this case is:

$$\begin{aligned} u_2(x_1, x_2) &= \begin{cases} v - x_1 + \mathbf{1}\{w_2 - x_1 > 1\}(v - \min\{v, 1\}), & x_2 > x_1, \\ \mathbf{1}\{w_2 > 1 - x_2\}(v - \min\{v, 1 - x_2\}), & x_2 \leq x_1, \end{cases} \\ &= \begin{cases} 2v - x_1 - \min\{v, 1\}, & x_2 > x_1, \\ v - \min\{v, 1 - x_2\} & x_2 \leq x_1. \end{cases} \end{aligned}$$

Clearly, the second case is maximized when $x_2 = x_1$. If $v \leq 1 - x_2$, then $2v - x_1 - \min\{1, v\} \geq v - \min\{v, 1 - x_2\} = 0$. Also, if $x_2 = x_1 \leq v/2$, then

$$\begin{aligned} v - 1 + x_1 &\leq v - 1 + v - x_1 \\ &= 2v - 1 - x_1, \\ &\leq 2v - x_1 - \min\{v, 1\}. \end{aligned}$$

	$W_2 \le x_1$	$W_2 > x_1$			
		$W_2 \le 1 + x_1$		$W_2 > 1 + x_1$	
		$x_1 \le 1/2$	$x_1 > 1/2$	$x_1 \le v/2$ or $x_1 \le 1 - v$	$x_1 > v/2$ and $x_1 > 1 - v$
Adversary strategy	W_2	W_2	x_1	W_2	x_1
Adversary payoff	$\gamma(v + W_2 - 1)$	$v - x_1$	$\gamma(v + x_1 - 1)$	$2v - 1 - x_1$	$v - 1 + x_1$
Player payoff	$(2 - \gamma)(v - W_2)$	$\gamma(v - W_2 + x_1)$	$v - x_1$	0	$v - x_1$
γ	$\mathbf{1}\{W_2 \ge 1/2\}$	$\mathbf{1}\{W_2 - x_1 \le v\}$	$\mathbf{1}\{W_2 - x_1 \le v\}$	n/a	n/a

Figure 2: Optimal adversary strategies and corresponding profit functions

So the adversary should bid greater than x_1, and will receive a profit of:

$$u_2(x_1, x_2) = 2v - 1 - x_1. \tag{15}$$

The player receives a profit of

$$u_1(x_1, x_2) = 0. \tag{16}$$

(e) $w_2 > 1 + x_1, x_1 > v/2$ The profit function in this case is:

$$\begin{aligned} u_2(x_1, x_2) &= \begin{cases} v - x_1 + \mathbf{1}\{w_2 - x_1 > 1\}(v - \min\{v, 1\}), & x_2 > x_1, \\ \mathbf{1}\{w_2 > 1 - x_2\}(v - \min\{v, 1 - x_2\}), & x_2 \le x_1, \end{cases} \\ &= \begin{cases} 2v - x_1 - \min\{v, 1\}, & x_2 > x_1, \\ v - \min\{v, 1 - x_2\}, & x_2 \le x_1. \end{cases} \end{aligned}$$

Clearly, the second case is maximized when $x_2 = x_1$. If $v = \min\{v, 1 - x_2\} = \min\{1, 1 - x_1\}$, then $2v - x_1 - \min\{v, 1\} \ge 0 = v - \min\{v, 1 - x_2\}$ and the best adversary strategy, profit, and corresponding player profit is as in Case (4). However, if $1 - x_1 < v$, then

$$\begin{aligned} v - \min\{v, 1 - x_1\} &= v - 1 + x_1, \\ &= v - 1 + 2x_1 - x_1, \\ &> 2v - 1 - x_1. \end{aligned}$$

Thus, the adversary should bid $x_2 = x_1$ which results in a profit of

$$u_2(x_1, x_1) = v - 1 + x_1. \tag{17}$$

The player then receives a profit of

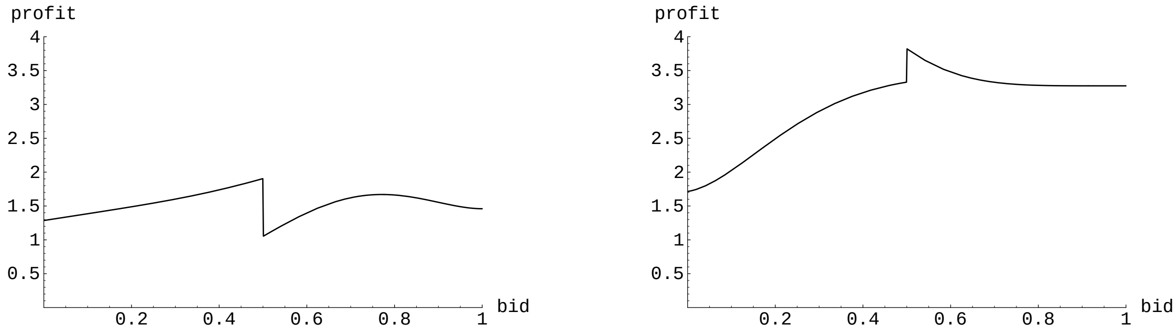
$$u_1(x_1, x_1) = v - x_1. \tag{18}$$

■

Figure 2 summarizes the weakly dominant adversary bids and the player and adversary profits in the five different cases.

Theorem 1 specifies the exact bid that the adversary used as a function of his budget, the player's bid, and the value of the objects, which results in the following Corollary.

Corollary 1 *The adversary gains nothing by using randomization to determine his round one bid.*



(a) $W_2 \sim Beta(5,2)$ (b) $W_2 \sim Beta(2,5)$

Figure 3: Player expected profit versus first round bid with common value $v = 2$

Theorem 1 also allows us to define a strategy function for the adversary with respect to the player's bid and his budget. Thus, let:

$$s_2 : [0,1] \times [0,\omega] \to [0,\omega], \tag{19}$$

be a function which indicates the adversary's bid under Theorem 1 according to the player's bid and the adversary's budget.

We now understand the adversary's best strategy, given what budget he ends up with. In order to analyze the player's best strategy, we must use her expected profit with respect to the adversary's budget. Assuming that the value of the object has been fixed, we can write both the player's and adverary's expected profit as a function of the player's bid.

$$m_i(x_1) := \mathbb{E}_{W_2}[u_i(x_1, s_2(x_1, W_2))] \tag{20}$$

To motivate what the best strategy will be for the player in this model, we examine some examples. In Figure 3(a), the best strategy for the player can be seen to bid $x_1 = 1/2$. In Figure 3(b), the best player strategy seems to be $x_1 = 1/2$, but because of the way in which ties are resolved, the actual best strategy is only realizable in an approximate sense. This motivates the following weakened definition of dominance for our model:

Definition 1 (Weakly limit-dominant strategy) *Let $m : \mathcal{X} \to \mathbb{R}$ be some given function representing the expected profit of a bidder playing a strategy from $\mathcal{X}$. $\bar{x}$ is a weakly limit-dominant strategy for the player, if, for all $\epsilon > 0$ there exists a $\delta > 0$ and strategy x such that*

$$|\bar{x} - x| < \delta, \tag{21}$$

and, for all $y \in \mathcal{X}$,

$$m(x) + \epsilon \geq m(y). \tag{22}$$

Note that, in our model, $m \equiv m_1$ and the strategy space is the budget set for player one, i.e., $\mathcal{X} = [0,1]$.

Then, for object values greater or equal to two (i.e., twice the player's budget), and any adversary budget distributions on $[0, 3/2]$, we can calculate either a weakly dominant or a weakly limit-dominant strategy.

Theorem 2 *Given that $v \geq 2$, and $\mathbb{P}$ is any probability measure on $[0, 3/2]$, with associated distribution function,m F, $x_1 = 1/2$ is either a weakly dominant strategy or a weakly limit-dominant stratey.*

Proof: To show this theorem, we will show that $x_1 = 1/2$ is weakly dominant on $[0, 1/2]$, and weakly limit-dominant on $(1/2, 1]$. Since a weakly dominant point is trivially weakly limit-dominant, the theorem follows.

We will first show that, for $x \in [0, 1/2)$,

$$m_1(1/2) \geq m_1(x).$$

To see this, choose some $x \in [0, 1/2)$, and note that, by Figure 2,

$$\begin{aligned}
m_1(1/2) &= \int_{[0,1/2]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2,3/2]} (v + 1/2 - w)\mathrm{d}\mathbb{P}(w) \\
&= \int_{[0,x]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{[x,1/2]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2,3/2]} (v + 1/2 - w)\mathrm{d}\mathbb{P}(w) \\
&\geq \int_{[0,x]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{[x,1/2]} (v + x - w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2,3/2]} (v + x - w)\mathrm{d}\mathbb{P}(w) \quad \text{(a)} \\
&= m_1(x).
\end{aligned}$$

Note that (a) is true since $v - w \geq 1/2 > x$ for all $w \in [0, 3/2]$, as $v \geq 2$, whence

$$\begin{aligned}
2(v-w) &= v - w + (v - w), \\
&> v - w + x.
\end{aligned}$$

Also note that the inequality in (a) is not strict since $\mathbb{P}$ may have no mass on $[x, 3/2]$ for some value $x \in [0, 1/2)$.

We will now show that $x_1 = 1/2$ is weakly limit-dominant on $(1/2, 1]$. First, for $\epsilon \in (0, 1/2)$, and $x \in (1/2+\epsilon, 1]$, we claim that

$$m_1(1/2 + \epsilon) \geq m_1(x).$$

To see this, choose some $x \in (1/2 + \epsilon, 1]$, and note that

$$\begin{aligned}
m_1(1/2+\epsilon) &= \int_{[0,1/2]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2,1/2+\epsilon]} (v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2+\epsilon,3/2]} (v - 1/2 - \epsilon)\mathrm{d}\mathbb{P}(w) \\
&= \int_{[0,1/2]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2,1/2+\epsilon]} (v-w)\mathrm{d}\mathbb{P}(w) \\
&\quad + \int_{(1/2+\epsilon,x]} (v - 1/2 - \epsilon)\mathrm{d}\mathbb{P}(w) + \int_{(x,3/2]} (v - 1/2 - \epsilon)\mathrm{d}\mathbb{P}(w) \\
&\geq \int_{[0,1/2]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2,1/2+\epsilon]} (v-w)\mathrm{d}\mathbb{P}(w) \\
&\quad + \int_{(1/2+\epsilon,x]} (v-w)\mathrm{d}\mathbb{P}(w) + \int_{(x,3/2]} (v-x)\mathrm{d}\mathbb{P}(w) \quad \text{(b)} \\
&= m_1(x).
\end{aligned}$$

Note that (b) follows since $v - (1/2 + \epsilon) > v - w$ for $w > 1/2 + \epsilon$, and, as before, the inequality used is not strict in case $\mathbb{P}$ has no mass on $(1/2 + \epsilon, 3/2]$.

We finally show that, for $x \in (1/2, 1/2 + \epsilon)$,

$$m_1(1/2 + \epsilon) + \epsilon \geq m_1(x, W_2).$$

To see this, choose some $x \in (1/2, 1/2 + \epsilon)$, and note that

$$\begin{aligned}
m_1(1/2+\epsilon) &= \int_{[0,1/2]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2,1/2+\epsilon]} (v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2+\epsilon,3/2]} (v - 1/2 - \epsilon)\mathrm{d}\mathbb{P}(w) \\
&= \int_{[0,1/2]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2,x]} (v-w)\mathrm{d}\mathbb{P}(w) + \int_{(x,1/2+\epsilon]} (v - x + x - w)\mathrm{d}\mathbb{P}(w) \\
&\quad + \int_{(1/2+\epsilon,3/2]} (v - x + x - (1/2+\epsilon))\mathrm{d}\mathbb{P}(w) \\
&\geq \int_{[0,1/2]} 2(v-w)\mathrm{d}\mathbb{P}(w) + \int_{(1/2,x]} (v-w)\mathrm{d}\mathbb{P}(w) + \int_{(x,1/2+\epsilon]} (v-x)\mathrm{d}\mathbb{P}(w) \\
&\quad - \epsilon \int_{(x,1/2+\epsilon]} \mathrm{d}\mathbb{P}(w) - \epsilon \int_{(1/2+\epsilon,3/2]} \mathrm{d}\mathbb{P}(w) + \int_{(1/2+\epsilon,3/2]} (v-x)\mathrm{d}\mathbb{P}(w) \quad \text{(c)} \\
&\geq m_1(x) - \epsilon.
\end{aligned}$$

Note that (c) follows since $x - w \geq -\epsilon$ for $w \in [x, 1/2+\epsilon]$ by the choice of x. Then, $x_1 = 1/2$ is weakly limit-dominant on $(1/2, 1]$, where, for a given ϵ, $\delta = 2\epsilon$, and $x = 1/2 + \epsilon$ can be used. ■

A naive characterization of when $x_1 = 1/2$ is weakly dominant as opposed to weakly limit-dominant is to determine if

$$m_1(1/2) \geq \lim_{\epsilon \rightarrow 0_+} m_1(1/2+\epsilon). \tag{23}$$

If (23) is satisfied, then $x_1 = 1/2$ is weakly dominant. Otherwise it is weakly limit-dominant. For continuous distributions on $[0, 3/2]$, there is also another, less obvious characterization:

Corollary 2 *Suppose that W is distributed continuously on $[0, 3/2]$. Then,*

$$\mathbb{P}\left[W \in (1/2, 3/2]\right] \mathbb{E}\left[W | W \in (1/2, 3/2]\right] = \int_{(1/2,3/2]} w \mathrm{d}\mathbb{P} \leq \mathbb{P}\left[W \in (1/2, 3/2]\right], \tag{24}$$

if and only if $x_1 = 1/2$ is a weakly dominant strategy.

Proof: Note first that:

$$m(1/2) = \int_{[0,1/2]} 2(v-w)\mathrm{d}\mathbb{P} + \int_{(1/2,3/2]} (v - w + 1/2)\mathrm{d}\mathbb{P},$$

and

$$m(1/2+\epsilon) = \int_{[0,1/2]} 2(v-w)\mathrm{d}\mathbb{P} + \int_{[1/2,1/2+\epsilon]} (v-w)\mathrm{d}\mathbb{P} + \int_{(1/2+\epsilon,3/2]} (v - 1/2 - \epsilon)\mathrm{d}\mathbb{P}.$$

We will first show the contrapositive of the forward direction. Suppose that 1/2 is not a weakly dominant strategy. Then, by Theorem 2, 1/2 is a weakly limit-dominant strategy, and for all $\epsilon > 0, m(1/2+\epsilon) + \epsilon - m(1/2) \geq 0$. So then,

$$\epsilon + m(1/2+\epsilon) - m(1/2) = \epsilon + \int_{(1/2+\epsilon,3/2]} (w - 1 - \epsilon)\mathrm{d}\mathbb{P} - \int_{[1/2,1/2+\epsilon]} (1/2)\mathrm{d}\mathbb{P} \geq 0.$$

Taking the limit as ϵ goes to zero shows that

$$\int_{(1/2,3/2]} (w-1)\mathrm{d}\mathbb{P} > 0.$$

the converse of 24.

Now uppose that 1/2 is a weakly dominant strategy, which implies that $m_1(1/2) \geq m_1(1/2+\epsilon)$ for all $\epsilon \in [-1/2, 1/2]$. Then, for some fixed $\epsilon > 0$,

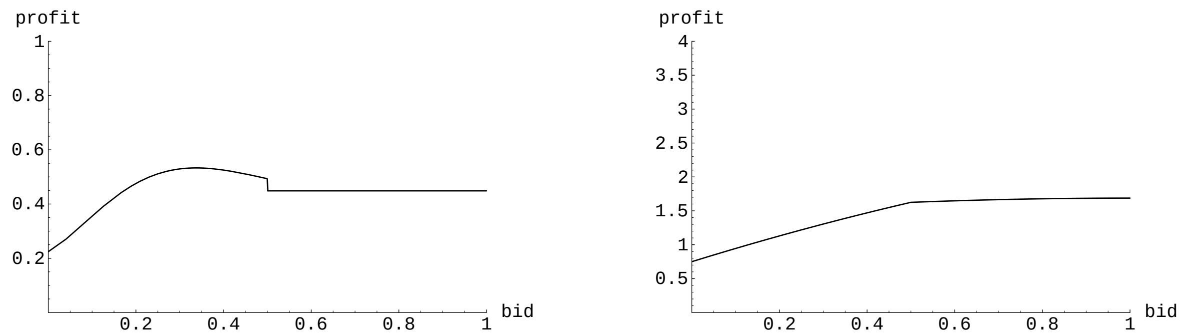
$$m(1/2) - m(1/2+\epsilon) = \int_{[1/2,1/2+\epsilon]} (1/2)\mathrm{d}\mathbb{P} + \int_{(1/2+\epsilon,3/2]} (1+\epsilon-w)\mathrm{d}\mathbb{P} \geq 0,$$

but taking limits as ϵ goes to zero yields (24).

■

We note that Theorem 2 does not hold for distributions where $W_2 > 3/2$ or when the value $v < 2$. The former can be seen in Figure 4(b), and the latter in Figure 4(a). We present some examples of Equation (7) applied to different distributions and common values the appendix.

Here is another example for which Theorem 2 does not hold. Let $v = 2$, and let W_2 be uniformly distributed on $[1/2, 3/2]$ with probability 1/2, and set to 7/4 with probability 1/2. We now show that the best player strategy if the adversary has this distribution of budget is to bid $x_1 = 3/4$.



(a) $W_2 \sim Beta(2,5), v = 0.5$ (b) $W_2 \sim U[0,2], v = 2$

Figure 4: Cases not covered by Theorem 1

Under this distribution, the profit for the player when bidding $x < 1/2$ is

$$m_1(x) = 1/2 \int_{w=1/2}^{1+x} (2 - w + x) dw = 5/16 + 3x/4 + x^2/4. \quad (25)$$

This is a linear function which is increasing in x, and thus assumes its supremum at $x = 1/2$. Note that $\sup_{x<1/2} m_1(x) = 3/4$. The profit for the player bidding $x \in [1/2, 3/4)$ is

$$m_1(x) = 1/2 \big[\int_{w=1/2}^{x} (2 - w) dw + \int_{w=x}^{3/2} (2 - x) dw \big] = 17/16 - 3x/4 + x^2/4. \quad (26)$$

This is a concave function with minimum at $x = 3$. Since this function is only defined along the interval $[1/2, 3/4)$, it attains its supremum at 3/4 with $\sup_{x \in [1/2,3/4)} m(3/4) = 41/64$. Note that this strategy is clearly suboptimal, since the player will lose both rounds with probability 1/2, i.e., when the adversary is endowed with a budget of 7/4. By this logic, the best strategy should be when the player bids $x \in [3/4, 1]$. The profit function in this case is

$$m_1(x) = 1/2 \big[\int_{w=1/2}^{x} (2 - w) dw + \int_{w=x}^{3/2} (2 - x) dw \big] + 1/2(2 - x) = 33/16 - 5x/4 + x^2/4. \quad (27)$$

This function is concave with a minimum at $x = 5/2$. Thus, since it occurs on a closed interval, its maximum exists on this interval at $x = 3/4$, and has value $m(3/4) = 81/64$. This is the best strategy for the player, and corresponds to the smallest amount necessary to win a round if the adversary is rich. The amount should be as small as possible to minimize the impact of jamming from the adversary.

4 The second model

As with the first model, we describe the second model from the perspective of the sequence of bidding and of information availability. The second model generalizes the first model by allowing the player to bid a probabilistic strategy.

1. The player reveals a distribution, Σ_1, on her strategy space, $[0, 1]$.

2. With the knowledge of Σ_1, the adversary bids a distribution, Σ_2, on his strategy space, $[0, \omega]$.

	Round	Information known prior to bidding	Hidden Information*	Action
Player	1	$v, W_1 = 1, F$	x_1, x_2, W_2	Σ_1, dist. on $[0, 1]$
	2	x_1, x_2, r_1	x_2^2, r_2	
Adversary	1	$v, W_1 = 1, W_2, \Sigma_1$	x_1, x_2	Σ_2, dist. on $[0, W_2]$
	2	$x_1, x_2, v, r_1, r_2, x_1^2$		

*note that x_i are not hidden to bidder i if Σ_i is a point-mass

Figure 5: Model 2

3. Two random draws are generated independently from Σ_1 and Σ_2. The higher bidder wins paying the lower bid (second-price).

Note that the two distributions are independent. Also, Lemma 1 continues to hold, so the second round bids are determined by the remaining budgets and common value, v.

From the perspective of the availability of information, the model can be described by Figure 5: The information revealed to each bidder is not "forgotten," so in round 2, each bidder knows everything from round 1.

We let S_i denote the random variable associated with the distribution Σ_i. Then define

$$m_2(x_2, \Sigma_1) = \mathbb{E}[u_2(S_1, S_2)|S_2 = x_2]. \tag{28}$$

When it is clear what Σ_1 is, we will simply write $m_2(x)$.

4.1 Uniform strategies

In this section, we consider the case where Σ_1 is distributed uniformly on an interval $[a, b]$ where $1/2 \leq a < b \leq 1$ and $b > 1/2$.

First, we will examine the optimal adversary strategies.

Theorem 3 *Suppose the player chooses her first bid according to Σ_1 for given a and b. Then a weakly limit-dominant strategy for the adversary with a budget of w_2 is:*

$$x_2 := \begin{cases} w_2 & w_2 < \frac{b+1}{3}, \\ \max\{a, \frac{b+1}{3}\} & w_2 \in (\frac{b+1}{3}, 1+a], \\ & \textit{or, } w_2 \in [1+a, 1+b), w_2 - 1 < \frac{b+1}{3}, \\ w_2 - 1 & w_2 \in [1+a, 1+b), w_2 - 1 \geq \frac{b+1}{3}, \\ w_2 & \textit{otherwise.} \end{cases} \tag{29}$$

Moreover, the strategy is weakly dominant for all cases except when $w_2 \in [1 + a, 1 + b)$ and $w_2 - 1 \geq \frac{b+1}{3}$.

Proof: Throughout this proof, we will use the variable y to denote a sample of the player's first bid. Note first that if the adversary's budget is less than or equal to a, his best strategy is to bid his budget in the hopes of winning the second round. Otherwise, his profit function is:

$$m_2(x) := \frac{1}{b-a}\Big[\int_a^x (v - y + \mathbf{1}\{w_2 - y > 1\}(v-1))dy + \int_x^b \mathbf{1}\{w_2 > 1 - x\}(v - 1 + x)dy\Big].$$

We examine the four cases, (1) $a < w_2 \leq \frac{b+1}{3}$, (2) $1 + a \geq w_2 > \max\{a, \frac{b+1}{3}\}$, (3) $1 + b \geq w_2 > 1 + a$, and (4) $w_2 > 1 + b$.

(1) $a < w_2 \le \frac{b+1}{3}$.
The adversary could either bid $x_2 \in [a, \min\{w_2, 1-w_2\}]$ or $x_2 > \min\{w_2, 1-w_2\}$. In the former case,

$$m_2(x_2) = \frac{1}{b-a}\int_a^{x_2}(v-y)dy = \frac{1}{b-a}(v(x_2-a) - (x_2^2 - a^2)/2).$$

This is a concave function with maximum at $x_2 = v$. Thus, on the interval $[a, \min\{w_2, 1-w_2\}]$, the best strategy is for the adversary as much as possible, $x_2 = \min\{w_2, 1-w_2\}$. Note that this is the sole case when $w_2 \le 1 - w_2$.

So, suppose that $w_2 > 1 - w_2$ and that the adversary bids $x_2 > 1 - w_2$. Then,

$$\begin{aligned} m_2(x_2) &= \tfrac{1}{b-a}\big[\textstyle\int_a^{x_2}(v-y)dy + \int_{x_2}^{b}(v-1+x_2)dy\big], \\ &= \tfrac{1}{b-a}\big[v(x_2-a) - (x_2^2-a^2)/2 + (v-1+x_2)(b-x_2)\big], \\ &= \tfrac{1}{b-a}\big[-\tfrac{3}{2}x_2^2 + (b+1)x_2 + (v(b-a) - a^2/2 - b)\big]. \end{aligned}$$

This is concave with maximum at $x_2 = \frac{b+1}{3}$, and the best strategy for the adversary when restricted to the interval $(1-w_2, w_2]$ is to bid w_2. We now must show that this strategy is better when x_2 is set to $1-w_2$, which we saw is the best strategy when the adversary restricts his bid to the interval $[0, 1-w_2]$. First, note that $b > 1/2$, so $b = b/3 + 2b/3 > b/3 + 1/3$, whence $b > x_2$ and

$$(v - 1 + x_2)(b - x_2) > 0. \tag{30}$$

Moreover, for $x_2 \in (1-w_2, w_2]$ and $u < 1-w_2$, $x_2 + u \le 1 < v$, whence $v(x-u) > (x+u)(x-u) = x^2 - u^2$. This implies that $v(x_2-a) - (x_2^2 - a^2)/2 > (v(u-a) - (u_2^2 - a^2)/2$. Thus, $m_2(x_2) > m_2(u)$ for $x_2 \in (1-w_2, w_2]$ and $u < 1-w_2$. So the best strategy for the adversary is to bid $x_2 = w_2$.

(2) $w_2 \in (\max\{a, \frac{b+1}{3}\}, 1+a)$.
The analysis is actually identical to the previous case, except that bidding $x_2 \ge \frac{b+1}{3}$ is possible. Note that the analysis assumes the function is restricted to $w_2 \ge a$. Thus, bidding $x_2 = \max\{a, (b+1)/3\}$ is the best strategy for the adversary.

(3) $1 + b \ge w_2 > 1 + a$ In this case, if the adversary bids $x \in (a, w_2 - 1]$, he receives

$$\begin{aligned} m_2(x) &= \tfrac{1}{b-a}\big[\textstyle\int_a^{x}(v-y+v-1)dy + \int_{x}^{b}(v-1+x)dy\big], \\ &= \tfrac{1}{b-a}\big[-\tfrac{3}{2}x^2 + (v+b)x + vb + a + a^2/2 - 2va - b\big]. \end{aligned}$$

This is concave and has a maximum of $(v+b)/3$ and, since $v > 2b$ implies that $w_2 - 1 < (v+b)/3$, is equivalent to the strategy of bidding $x_2 = w_2 - 1$.

Now suppose the adversary bids $x \in (w_2 - 1, b]$. Then, his profit is

$$\begin{aligned} m_2(x) &= \tfrac{1}{b-a}\big[\textstyle\int_a^{w_2-1}(v-y+v-1)dy + \int_{w_2-1}^{x}(v-y)dy + \int_{x}^{b}(v-1+x)dy\big], \\ &= \tfrac{1}{b-a}\big[(v-1)(w_2-1-a) + v(b-a) - (x^2-a^2)/2 + (b-x)(-1+x)\big], \\ &= \tfrac{1}{b-a}\big[-\tfrac{3}{2}x^2 + (b+1)x + a^2/2 + v(b-a) - b + a - va + (v-1)(w_2-1)\big]. \end{aligned}$$

This concave function has a maximum at $(b+1)/3 \le 2/3 < w_2$, if $(b+1)/3 > w_2 - 1$. If not, then the maximum doesn't exist, although the supremum is $w_2 - 1$.

We will now show that the latter strategy is either a dominant strategy, or a limit-dominant strategy. There are two cases to consider. Either the best bid in the interval $(w_2 - 1, b]$ is $(b+1)/3$ or, the best bid is to bid arbitrarily close to $w_2 - 1$. Bidding $w_2 - 1$ is the best strategy when bidding $(a, w_2 - 1]$. To see this, first consider the case when $w_2 - 1 \geq (b+1)/3$, and let $\epsilon > 0$ be some small quantity, $x_+ = w - 1 + \epsilon$, and $x = w - 1$. Then,

$$\lim_{\epsilon \to 0} m_2(x_+) - m_2(x) = v(w-1) - b(w-1) \geq 0.$$

Also, if $w_2 - 1 < (b+1)/3$, then let $u = (b+1)/3$ and, as before, $x = w - 1$. Then, some algebra shows that:

$$\begin{aligned} m_2(u) - m_2(x) &= -\tfrac{3}{2}u^2 + (b+1)u + (v-1)(w_2-1) + \tfrac{3}{2}x^2 - (v+b)x \\ &= -\tfrac{(b+1)^2}{6} + \tfrac{(b+1)^2}{3} + (v-1)(w_2-1) - (v+b)(w_2-1) + \tfrac{3}{2}(w-1)^2 \\ &= \tfrac{1}{6}\big[(b+1)^2 - 6(b+1)(w_2-1) + 9(w_2-1)^2, \\ &= \tfrac{1}{6}(b+1-3(w_2-1))^2 \geq 0. \end{aligned}$$

Thus, a weakly dominant strategy (possibly limit-dominant) is $x_2 = (b+1)/3$, if $(b+1)/3 > w_2 - 1$, and $x_2 = w_2 - 1$ for some $\epsilon > 0$ otherwise.

(4) $w_2 > 1 + b$
In this case, for a bid of $x \in [a, b]$, the adversary receives

$$\begin{aligned} m_2(x) &= \tfrac{1}{b-a}\big[\int_a^x (v - y + v - 1)dy + \int_x^b (v - 1 + x)dy\big], \\ &= \tfrac{1}{b-a}\big[-\tfrac{3}{2}x^2 + (v+b)x + (b-a)(v-1) - va + a^2/2\big]. \end{aligned}$$

Note that a bid of $x > b$ is equivalent to bidding $x = b$. This function is convex, and the maximum is obtained at $(v+b)/3 > b$, as $v > 2b$. Thus, the best strategy for the adversary is to bid greater than b.

■

Recall that the player in Model 1 corresponds to a player in Model 2 using a point-mass distribution on one point in the interval $[0, 1]$ as a strategy. We will refer to this as the fixed-point strategy. Note that the uniform strategy will always dominate the fixed point, as the uniform strategy can duplicate this by setting $a = b$. Moreover, there are instances where the uniform strategy is strictly better than the fixed point strategy. Consider the following example: Suppose the adversary has a budget that is distributed uniformly on the interval $[1, 2]$, and that $v = 2$. The profit function for a player bidding a fixed-point strategy, x, is:

$$m_1(x) = \int_1^{1+x} (2 - x)dw = 2x - x^2.$$

This concave function is maximized at $x = 1$. Thus, the best strategy for the player is to bid $x = 1$, with $m(1) = 1$. Now consider the uniform strategy with $a = 1/2$ and $b = 1$. Then, according to Theorem 3, the adversary will bid:

$$x_2 = \begin{cases} 2/3 & w_2 \leq 5/3, \\ w_2 - 1 + \epsilon & \text{otherwise.} \end{cases}$$

For notational convenience, we will assume we are operating at the limit as $\epsilon \to 0$, and ignore the ϵ. Then, the player receives:

$$\begin{aligned} m_1(\Sigma) &= \int_1^2 \int_{1/2}^1 \big[\mathbf{1}\{w \leq 5/3\}(\mathbf{1}\{x < 2/3\}(2 - w + x) + \mathbf{1}\{x \geq 2/3\}(5/3)) \\ &+ \mathbf{1}\{w > 5/3\}\mathbf{1}\{x > w - 1\}(2 - (z - 1))\big]2dxdw, \\ &= 2 * \big[\int_1^{5/3}\int_{1/2}^{2/3}(2 - w + x)dxdw + \int_1^{5/3}\int_{2/3}^1 (5/3)dxdw + \int_{5/3}^1 \int_{w-1}^1 (3 - z)dxdw\big], \\ &= 15/54 + 40/54 + 11/81 = 187/162. \end{aligned}$$

The essential difference between the fixed-point strategy and the uniform strategy is that the adversary is no longer able to jam the player as successfully – any nontrivial bid the adversary makes has a positive probability of winning in the first round, causing the player to jam the adversary.

5 More than one object value

Consider the case when the objects auctioned in each round are no longer identical, and instead have values v_i in rounds $i = 1, 2$. Note that, in this case, Lemma 1 still holds. However, the profit functions for the player and adversary have now changed to:

The player's profit is:

$$\begin{aligned} u_1(x_1, x_2) \quad := \quad & \mathbf{1}\{x_1 \geq x_2\}\big(v_1 - x_2 + \mathbf{1}\{1 - x_2 \geq W_2\}(v_2 - W_2)\big) \\ & + \mathbf{1}\{x_1 < x_2\}\mathbf{1}\{W_2 - x_1 \leq 1\}(v_2 - W_2 + x_1) \end{aligned} \tag{31}$$

The adversary's profit is:

$$\begin{aligned} u_2(x_1, x_2) \quad := \quad & \mathbf{1}\{x_1 < x_2\}\big(v_1 - x_1 + \mathbf{1}\{W_2 - x_1 \geq 1\}(v_2 - 1)\big) \\ & + \mathbf{1}\{x_1 \geq x_2\}\mathbf{1}\{1 - x_2 \leq W_2\}(v_2 - 1 + x_2) \end{aligned} \tag{32}$$

For certain item, adversarial budget and player bid values, the strategy of the adversary will shift from winning the first auction to jamming the first auction. The "critical value," first described by Pitchik [13], can be expressed as:

$$\bar{c}(v_1, v_2) = \frac{1 + v_1 - v_2}{2}. \tag{33}$$

We will focus on this function defined only along the player's feasible bidding region, or,

$$c(v_1, v_2) = \max\big\{\min\{c(v_1, v_2), 1\}, 0\big\} \tag{34}$$

Note that in the case that $v_1 = v_2, c(v_1, v_2) = 1/2$.

Theorem 4 *Given that the player bids x_1 in round 1,a weakly dominant strategy for the adversary is to bid*

$$x_2 := \left\{ \begin{array}{ll} w_2 & \begin{array}{l} w_2 < x_1, \\ x_1 < w_2 \leq 1 - x_1, \\ \max\{x_1, 1 - x_1\} < w_2 \leq 1 + x_1, x_1 \leq c(v_1, v_2), \\ w_2 > 1 + x_1, \text{ and } (x_1 \leq v/2 \text{ or } \ v_2 \leq 1); \end{array} \\ \hline x_1 & \begin{array}{l} \max\{x_1, 1 - x_1\} < w_2 \leq 1 + x_1, x_1 > c(v_1, v_2), \\ w_2 > 1 + x_1, x_1 > v_1/2, \text{ and } \ v_2 > 1. \end{array} \end{array} \right. \tag{35}$$

Proof: We first note that there is never any incentive for the player to choose $x_1 > v_1$. If she did so, then the adversary would bid x_1 as well, causing the player to incur negative profit, and lose budget for the second round. By bidding a smaller amount, the player would be able to decrease the negative profit incurred and increase the chance of winning in the second round. We therefore assume that

$$x_1 \leq \min\{1, v_1\}. \tag{36}$$

We will prove the theorem using case analysis.

(a) $w_2 \le x_1$
By (36), $w_2 \le x_1 \le v_1$. Consider the profit function:

$$\begin{aligned} u_2(x_1, x_2) &= \mathbf{1}\{1 - x_2 < w_2\}\mathbf{1}\{1 - x_2 < v_2\}(v_2 - (1 - x_2)), \\ &= \begin{cases} v_2 + x_2 - 1, & \min\{v_2, w_2\} + x_2 > 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{37}$$

This is clearly maximized when the adversary bids as high as possible, i.e., $x_2 = w_2$. In this case, the player's profit is:

$$u_1(x_1, w_2) = v_1 - w_2 + \mathbf{1}\{w_2 + \min\{w_2, v_2\} \le 1\})(v_2 - w_2). \tag{38}$$

(b) $w_2 \in (x_1, 1 - x_1]$
Note that this case is only possible when $x_1 < 1 - x_1$. Moreover, the adversary will receive a profit of zero by losing the first round in this case, since the player will be able to win the second round if the adversary bids $x_2 \le x_1$. So, the best strategy is for the adversary to bid $x_2 \in (x_1, w_2]$. Note that this still may result in the adversary gaining nothing if $v_1 = x_1$. The profit received from the adversary under this strategy is:

$$u_2(x_1, x_2) = v_1 - x_1. \tag{39}$$

The player will receive

$$u_1(x_1, x_2) = v_2 - w_2 + x_1. \tag{40}$$

(c) $\max\{x_1, 1 - x_1\} < w_2 \le 1 + x_1, x_1 \le c(v_1, v_2)$
In this case, the profit function is:

$$\begin{aligned} u_2(x_1, x_2) &= \begin{cases} v_1 - x_1 + \mathbf{1}\{w_2 - x_1 > 1\}(v_2 - \min\{v_2, 1\}), & x_2 > x_1, \\ \mathbf{1}\{w_2 > 1 - x_2\}\mathbf{1}\{v_2 > 1 - x_2\}(v_2 - 1 + x_2), & x_2 \le x_1, \end{cases} \\ &= \begin{cases} v_1 - x_1, & x_2 > x_1, \\ \mathbf{1}\{w_2 + x_2 > 1\}\mathbf{1}\{v_2 > 1 - x_2\}(v_2 - 1 + x_2) & x_2 \le x_1. \end{cases} \end{aligned}$$

But, since $x_1 \le c(v_1, v_2)$,

$$\begin{aligned} v_1 - x_1 &\ge v - c(v_1, v_2), \\ &= v_1 - (1 + v_1 - v_2)/2, \\ &= v_2 - 1 + (v_1 - v_2 + 1)/2, \\ &= v_2 - 1 + c(v_1, v_2), \\ &\ge v_2 - 1 + x_1. \end{aligned}$$

This is an upper bound on the profit obtained by the adversary if he bids $x_2 \le x_1$. Thus, the adversary's best strategy is to bid some value greater than x_1 and he receives a profit of

$$u_2(x_1, x_2) = v_1 - x_1. \tag{41}$$

In this case, the player receives a profit of

$$u_1(x_1, x_2) = v_2 - \min\{v_2, w_2 - x_1\}. \tag{42}$$

(d) $\max\{1-x_1, x_1\} < w_2 \le 1+x_1, x_1 > c(v_1,v_2)$
This case is similar to the previous. The adversary's profit function is the same:

$$\begin{aligned} u_2(x_1,x_2) &= \begin{cases} v_1 - x_1 + \mathbf{1}\{w_2 - x_1 > 1\}(v_2 - \min\{v_2, 1\}), & x_2 > x_1, \\ \mathbf{1}\{w_2 > 1 - x_2\}\mathbf{1}\{v_2 > 1 - x_2\}(v_2 - 1 + x_2), & x_2 \le x_1, \end{cases} \\ &= \begin{cases} v_1 - x_1, & x_2 > x_1, \\ \mathbf{1}\{w_2 + x_2 > 1\}\mathbf{1}\{v_2 > 1 - x_2\}(v_2 - 1 + x_2) & x_2 \le x_1. \end{cases} \end{aligned}$$

However, now $x_1 > c(v_1,v_2)$, so:

$$\begin{aligned} v_1 - x_1 &< v - c(v_1,v_2), \\ &= v_1 - (1 + v_1 - v_2)/2, \\ &= v_2 - 1 + (v_1 - v_2 + 1)/2, \\ &= v_2 - 1 + c(v_1,v_2), \\ &< v_2 - 1 + x_1, \end{aligned}$$

and the adversary should bid $x_2 = x_1$. His profit is

$$u_2(x_1,x_1) = v_2 - \min\{v_2, 1 - x_1\}. \tag{43}$$

The player receives a profit of

$$u_1(x_1,x_1) = v_1 - x_1. \tag{44}$$

(e) $w_2 > 1 + x_1, x_1 \le v_1/2$ or $v_2 \le 1$. The profit function in this case is:

$$\begin{aligned} u_2(x_1,x_2) &= \begin{cases} v_1 - x_1 + \mathbf{1}\{w_2 - x_1 > 1\}(v_2 - \min\{v_2, 1\}), & x_2 > x_1, \\ \mathbf{1}\{w_2 > 1 - x_2\}(v_2 - \min\{v_2, 1 - x_2\}), & x_2 \le x_1, \end{cases} \\ &= \begin{cases} v_1 + v_2 - x_1 - \min\{v_2, 1\}, & x_2 > x_1, \\ v_2 - \min\{v_2, 1 - x_2\} & x_2 \le x_1. \end{cases} \end{aligned}$$

But note that:

$$\begin{aligned} v_1 + v_2 - x_1 - \min\{1, v_2\} &\ge v_1 + v_2 - 1 - x_1 \\ &\ge v_1 + v_2 - 1 - v_1/2 \\ &= v_2 - 1 + v_1/2 \\ &\ge v_2 - 1 + x_1 \\ &\ge v_2 - 1 + x_2. \end{aligned}$$

So the adversary should bid greater than x_1, and will receive a profit of:

$$u_2(x_1,x_2) = v_1 + v_2 - x_1 - \min\{1, v_2\}. \tag{45}$$

The player receives a profit of

$$u_1(x_1,x_2) = 0. \tag{46}$$

(f) $w_2 > 1 + x_1, x_1 > v_1/2$ and $v_2 > 1$. The profit function in this case is:

$$\begin{aligned} u_2(x_1,x_2) &= \begin{cases} v_1 - x_1 + \mathbf{1}\{w_2 - x_1 > 1\}(v_2 - \min\{v_2, 1\}), & x_2 > x_1, \\ \mathbf{1}\{w_2 > 1 - x_2\}(v_2 - \min\{v_2, 1 - x_2\}), & x_2 \le x_1, \end{cases} \\ &= \begin{cases} v_1 + v_2 - x_1 - \min\{v_2, 1\}, & x_2 > x_1, \\ v_2 - \min\{v_2, 1 - x_2\} & x_2 \le x_1. \end{cases} \end{aligned}$$

But note that, since $v_2 > 1$,

$$\begin{aligned} v_1 + v_2 - x_1 - \min\{1, v_2\} &= v_1 + v_2 - 1 - x_1 \\ &< v_1 + v_2 - 1 - v_1/2 \\ &= v_2 - 1 + v_1/2 \\ &< v_2 - 1 + x_1. \end{aligned}$$

Thus, the adversary should bid $x_2 = x_1$ which results in a profit of

$$u_2(x_1, x_1) = v_2 - 1 + x_1. \tag{47}$$

The player then receives a profit of

$$u_1(x_1, x_1) = v_1 - x_1. \tag{48}$$

■

We first note that there does not exist a weakly dominant fixed bid that is budget-distribution independent. That is, suppose v_1 and v_2 are fixed so that $v_1 + v_2 > 1$ and $c(v_1, v_2) \in (0, 1)$. Then, for any fixed bid x_1, there exists a distribution on the adversary's budget

Theorem 5 *Suppose $v_1 + v_2 > 1, c := c(v_1, v_2) \in (0, 1)$ and $v_1 > c$. There is no fixed bid $x_1 \in [0, 1]$ that is weakly limit-dominant across distributions.*

Proof: We will examine three cases: (1) $x_1 = c$, (2) $x_1 < c$, and (3) $x_3 > c$.

$x_1 = c$
Suppose the adversary has a budget that is distributed on $(1 + c, 2]$, and note that the adversary can win both rounds if $x_1 = c$. The adversary will do this, as opposed to jamming since the profit he receives from jamming satisfies:

$$\begin{aligned} v_2 - 1 + c &= v_2 - 1 + (v_1 + 1 - v_2)/2 \\ &= v_1 + v_2 - 1 - v_1 + (v_1 + 1 - v_2)/2 \\ &= v_1 + v_2 - 1 - v_1/2 - 1/2 + v_2/2 + 1 - v_2 \\ &< v_1 + v_2 - 1 - c, \end{aligned}$$

which is the profit for the adversary if he wins both rounds. Since $v_1 > c$, the player bid of c is dominated by the bid $x \in (c, v_1)$, as $v_2 = v_1 + 1 - c > 1$, and the adversary cannot win in both rounds. No matter which round the adversary wins, the player receives positive profit.

$x_1 < c$
Suppose that the adversary has a budget that is $1 + x_1$ with probability one. Then, according to Theorem 4 a player bid of $c + \epsilon$ for $\epsilon = (v_1 - c)/2$ dominates x_1 since the player will win the first round and results in a profit of $v_1 - c > 0$.

$x_1 > c$
Suppose that the adversary has a budget of $1 + c$ with probability one. Then, according to Theorem 4, a player bid of $\bar{x}_1 \in (c, x_1)$ dominates x_1 as the player is jammed in both cases, but receives $x_1 - \bar{x}_1$ more profit.

■

Although bidding the critical value is clearly seen to be a non-dominant strategy in general, the player can bid the critical value in order to cause the adversary to bid his budget, rather than jamming.

Theorem 6 *For values v_1 and v_2 such that $c := c(v_1, v_2) \in [0, 1]$, a player bid of $x_1 = c$ causes the adversary to bid his budget, i.e., he will not jam the player.*

Proof: By 4, if $w_2 \le c$, the adversary always bids w_2. So assume that $w_2 > c$. In this case, we simply have to show that $c > v_1/2$ and $v_2 > 1$ cannot both be true. To see that this is so, note that $v_1/2 < c \Leftrightarrow$:

$$\begin{aligned} 0 &< c - v_1 \\ &= -v_2 + 1, \end{aligned}$$

whence $v_2 < 1$. ∎

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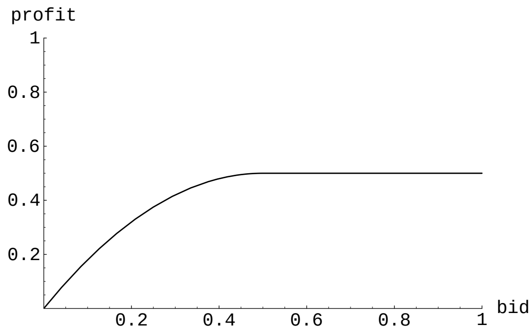
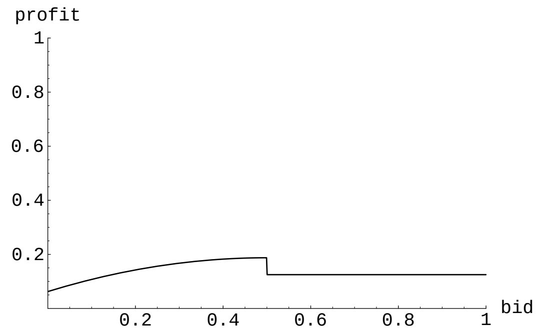
A Player expacted profits under different distributions

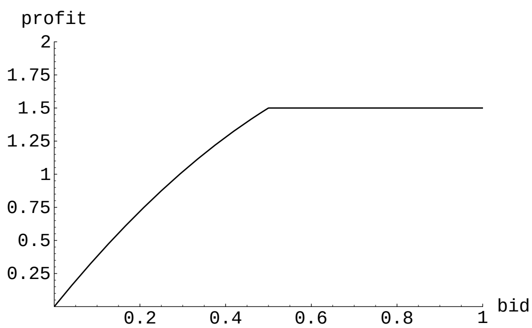


(a) $W_2 \sim U[0, 0.5]$

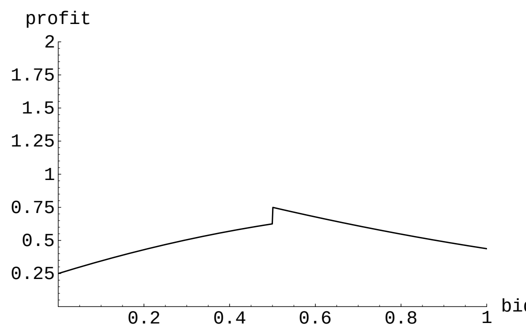


(b) $W_2 \sim U[0, 2]$

Figure 6: Common value $v = 0.5$

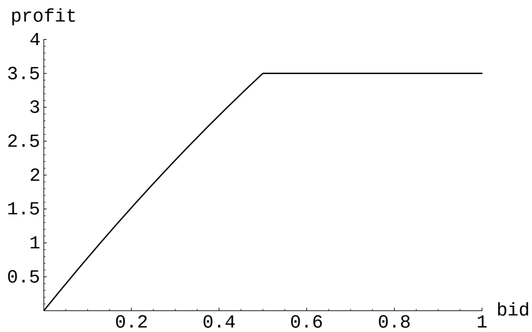


(a) $W_2 \sim U[0, 0.5]$

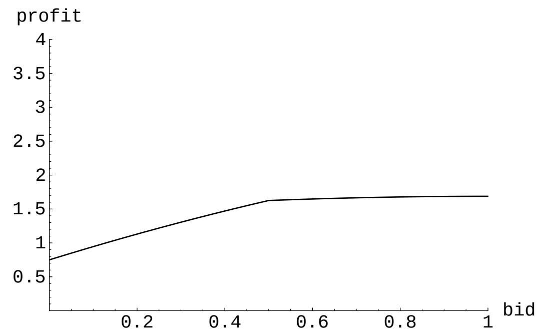


(b) $W_2 \sim U[0, 2]$

Figure 7: Common value $v = 1$

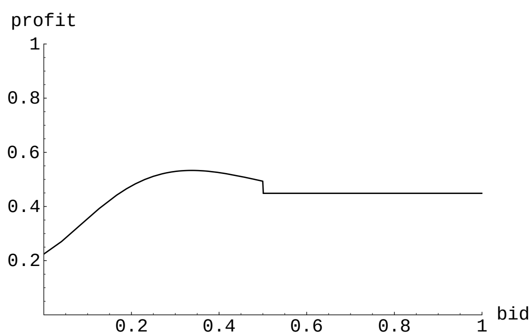


(a) $W_2 \sim U[0, 0.5]$

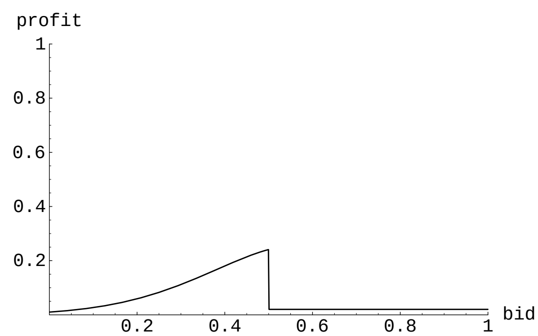


(b) $W_2 \sim U[0, 2]$

Figure 8: Common value $v = 2$

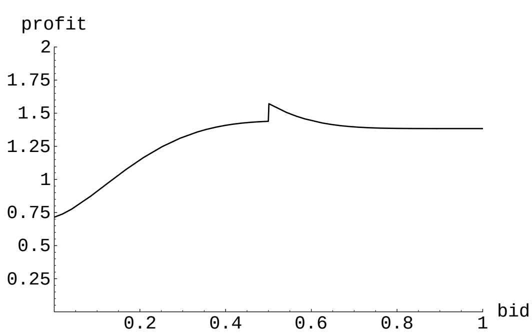


(a) $W_2 \sim Beta(2, 5)$

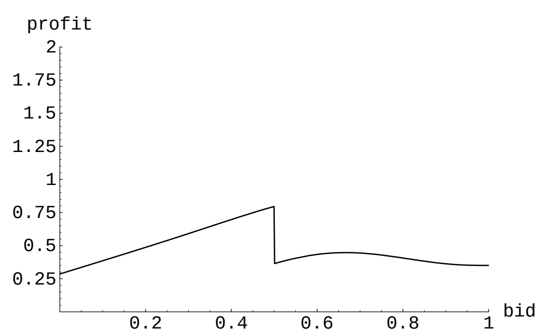


(b) $W_2 \sim Beta(5, 2)$

Figure 9: Common value $v = 0.5$

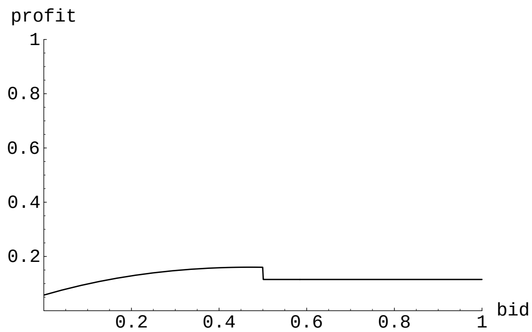


(a) $W_2 \sim Beta(2, 5)$

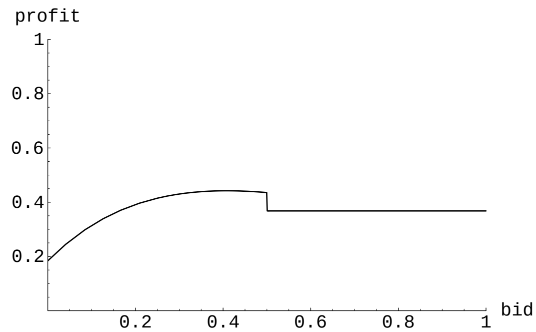


(b) $W_2 \sim Beta(5, 2)$

Figure 10: Common value $v = 1$

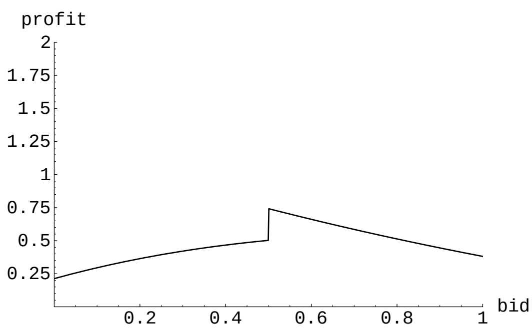


(a) $W_2 \sim Exp[0.5]$

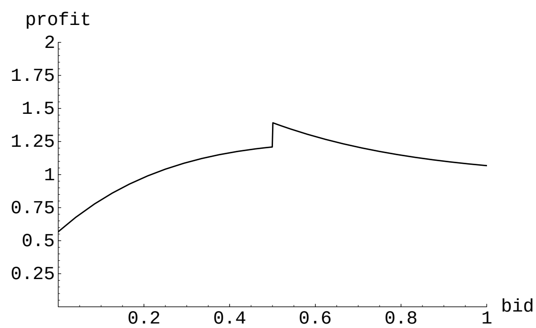


(b) $W_2 \sim Exp[2]$

Figure 11: Common value $v = 0.5$

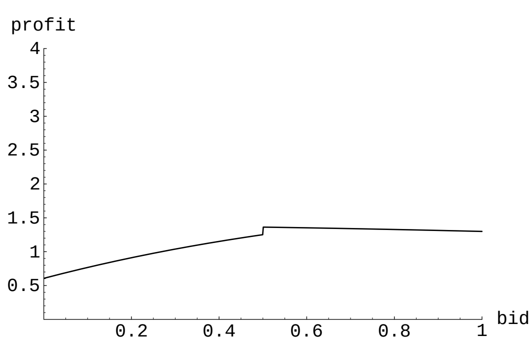


(a) $W_2 \sim Exp[0.5]$

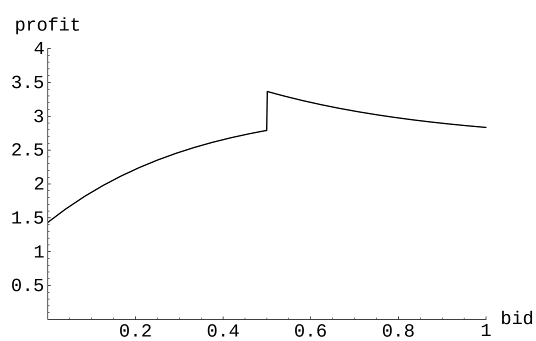


(b) $W_2 \sim Exp[2]$

Figure 12: Common value $v = 1$



(a) $W_2 \sim Exp[0.5]$



(b) $W_2 \sim Exp[2]$

Figure 13: Common value $v = 2$