Optimal Procurement Auctions of Divisible Goods with Capacitated Suppliers

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Abstract

The literature on procurement auctions (reverse auctions) typically assumes that the suppliers are uncapacitated (see, e.g. Dasgupta and Spulber, 1990; Ankolekar et al., 2005; Chen, 2004; Che, 1993). Consequently, these auction mechanisms award the contract to a single supplier. We consider a model where suppliers have limited production capacity, and both marginal costs and the production capacities are private information. We construct the optimal direct mechanism that maximizes the retailer's expected profit. We provide a closed-form solution when the distribution of the cost and production capacities satisfies a modified *regularity* condition (Myerson, 1981). We also present a sealed low bid implementation of the optimal direct mechanism for the special case of identical suppliers, i.e. symmetric environment. This implementation requires each supplier to submit a bid consisting of the desired marginal payment and total available production capacity. These bids serve as the input to a simple optimization problem that computes the quantity allocation for each firm. We extend the model to multi-product procurement with complementarities.

The results in this paper are applicable to a number of principle-agent mechanism design problems where the agents have privately known upper bound on the allocations. Examples of such problems include monopoly pricing with adverse selection and forward auctions.

Keywords: Reverse auction, Procurement auctions, Optimal direct mechanism, Capacity constraints, Multiple Sourcing.

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1 Background and Motivation

Awarding contracts via auctioning is now pervasive across many industries, e.g. electronics industry procurements, government defence procurements, and supply chain procurements. Since the auctioneer is the *buyer*, the bidders are the *suppliers* or *sellers*, and the object being auctioned is the right to supply, these auctions are also called *reverse auctions*. The use of reverse auctions to award contracts has been vigorously advocated since competitive bidding results in lower procurement costs, facilitates demand revelation, allows order quantities to be determined ex-post based on the bids and limits the influences of nepotism and political ties. Moreover, the advent of the Internet has significantly reduced the transaction costs involved in conducting such auctions. There is now a large body of literature detailing the growing importance of reverse auctions in industrial procurement. Parente et al. (2001) report that the total value of the B2B online auction transactions totaled 109 billion in 1999, and projected an excellent growth rate of the same.

Although auction design is a well-studied problem, the models analyzed thus far do not adequately address the fact that the private information of the bidders is typically multi-dimensional (cost, capacity, quality, lead times, etc.) and the instruments available to the buyer, i.e. the mechanism designer, to screen this private information is also multidimensional (multiple products, multiple components, different procurement locations, etc). This paper investigates mechanism design for a one-shot reverse auction with divisible goods and suppliers with finite capacity in single and multi-product environments. The production capacity, in addition to the production cost, are only known to the respective suppliers and need to screened by an appropriate mechanism. Thus in our model, the private information of the supplier is two dimensional.

The model in this paper is similar to Chen (2004) and Dasgupta and Spulber (1990), except for the fact that in our model the suppliers have finite production capacity. We refer to suppliers with finite capacity as *capacitated suppliers*.

This paper is organized as follows. We discuss the relevant literature in § 1.1. In § 2 we describe the model preliminaries. In § 3 we present the analysis for single product optimal direct auction mechanism and it's implementation via "pay as you bid" reverse auction. In § 4 we present a simple extension to the multi-product/component model, where the private information about the production cost of the supplier is modeled as one dimensional scalar quantity. In § 5 we discuss some of the limitation of the model and directions for future research.

1.1 Literature Review

The literature relevant to this problem can be primarily categorized into the following two categories.

1.1.1 Operations Management

There is some previous work on supply chain models with finite capacities. Benson (1995) investigates optimal allocation in a multi-component environment where the buyer is simply a price-taker, i.e. the buyer pays whatever the suppliers bid. It is easy to convince oneself that in such an environment the suppliers will distort their bids, resulting in very low profits for the buyer.

Swaminathan et al. (1995) study the effect of sharing supplier capacity information on the channel profit and profits to individual entities in a model with one manufacturer and two suppliers that differ in cost and capacity. They conclude that information sharing is beneficial to overall supply chain performance; however, it can be detrimental to individual suppliers. It follows, therefor, that unless the suppliers are given proper incentives they are unlikely to reveal their privately known capacity and production cost, and the predicted improvement in channel profits will not be realized. The model investigated in Gallien and Wein (2005) is similar to the multi-product model in this paper. They propose a multi-round mechanism that is neither incentive compatible nor is it optimal. It is difficult to justify that in the proposed multi-round mechanism, where the outcome is determined only in the final round, the suppliers do not have the incentive to deviate from the *myopic best response* (MBR) strategy. Also the results in Gallien and Wein (2005) rely on linear programming duality and complimentary slackness; therefore, they may not generalize to more general cost structures.

There is a growing body of work on applying mechanism design techniques to study decentralized decision making and contract design in supply chain management. Deshpande and Schwarz (2005) consider an asymmetric information model with single supplier and many retailers and the retailer's order are influenced by privately known demand. They design pricing and allocation (in case of shortage) mechanisms that ensure that the retailers reveal the demand information truthfully. Zhang (2005) considers a reverse auction in which the buyer's profit also depends on the lead time offered by the supplier. This paper constructs a mechanism in which the buyer discriminates between suppliers by both their posted lead time as well as the available inventory. However, only the marginal cost information is private, i.e. the supplier type is one dimensional. Chen et al. (2005) design a *Vickrey-Clark-Groves* (VCG) mechanism for a supply chain reverse auction with transportation costs. Since the profit to the principal in an efficient auction (i.e. a social welfare maximizing auction) can be arbitrarily smaller than the profit in a revenue maximizing auction, the principal has an incentive to distort information provided to the *third party* auctioneer. For this reason, Chen et al. (2005) provide three different auction formats to investigate the relative distortion of the information provided by the bidders and analyze its impact on realized channel profit. The model in this paper assumes that the transportation costs are common knowledge; thus, the agent type space is again one-dimensional. Using a number of simple models, Jin and Wu (2001) show that auctions are an effective mechanism for coordinating supply chains. Beil and Wein (2001) consider a manufacturer who uses a reverse auction to award a contract to a single supplier based on both prices and a set of non-price attributes that directly affect the valuations. Ankolekar et al. (2005) study the design of optimal supply contract when the buyer order is determined after the demand realization, but production as a function of winning supplier cost is determined before demand realization.

1.1.2 Microeconomic Theory

In this section we review the microeconomic theory literature on optimal mechanism design with multi-dimensional type or multi-dimensional screening instrument space, which is relevant to this work. Myerson (1981) first used the indirect utility approach to characterize the optimal auction in an *independent private value* (IPV) model. Che (1993) considers 2-dimensional (reverse) auction where the sellers bid price and quality and the principal's preference is over both quality and price. However, only the costs are private information and the quality preferences are common knowledge; thus, the bidder type space is one-dimensional. Also, Che (1993) only considers sourcing from a single supplier. This leads to a considerable simplification since it reduces the problem to one of determining the winning probability instead of the expected allocation. Naegelen (2002) models reverse auctions for department of defense (DoD) projects by a model where the quality of each of the firms are fixed and common knowledge. The preference over quality in this setting results in virtual utilities which are biased across suppliers. Again she only consider single winner case.

Dasgupta and Spulber (1990) consider a model very similar to the one discussed in this paper except that the suppliers have unlimited capacity. They construct the optimal auction mechanism for both single sourcing and multiple sourcing (due to non-linearities in production costs) when the private information is one-dimensional. Chen (2004) presents an alternate two-stage implementation for the optimal mechanism in Dasgupta and Spulber (1990). In this alternate implementation the winning firm is first determined via competition on fixed fees, and then the winner is offered an optimal price-quantity schedule.

Laffont et al. (1987) solve the optimal nonlinear pricing (single agent principal-agent mechanism design) problem with a two-dimensional type space. They explicitly force the integrability conditions on the gradient of the indirect utility function. Surprisingly, the optimal pricing mechanism (the bundle menus) is rather involved even in the simple setting with a *uniform* prior distribution. Rochet and Stole (2003) also provide an excellent survey of multi-dimensional screening and the associated difficulties. In appendix A, we discuss a reverse auction model with capacitated suppliers having convex quadratic costs, where both linear coefficient (representing scale) and the quadratic coefficient (representing capacity) are privately known. We describe the associated optimization problem to explain the complexities that arise with multi-dimensional types.

Vohra and Malakhov (2004) describes the indirect utility approach in discrete type-spaces. They re-derive many of the existing results for auctions using network flow techniques and consider optimal auctions with multidimensional types. In Vohra and Malakhov (2005), the authors use the same techniques with discrete types-space in an multi-unit optimal auction model where the bidders have privately know capacities in addition to the privately known marginal values. In contrast to Vohra and Malakhov (2005),

i) we consider *variable quantity reverse auctions* with *continuous type space*, which allows us to work with more general utility structures;

ii) characterize the set of all incentive compatible mechanisms without assuming monotone allocations;

iii) present an *ironing procedure* under which the optimal mechanism can be characterized under milder regularity conditions;

iv) present a low bid implementation of our optimal direct mechanism and

v) give extensions to capacitated multi-product model.

Voicu (2002) consider the procurement auction in a dynamic environment, where bidder takes in to account the possible outcomes of future auctions in a dynamic programming framework.

2 Procurement Auctions with Finite Supplier Capacities

We consider a single period model with one buyer (retailer, manufacturer, etc.) and n suppliers. The buyer purchases a single commodity from the suppliers and resells it in the consumer market. The buyer receives an expected revenue, $R(q)$ from selling q units of the product in the consumer market – the expectation is over the random demand realization and any other randomness involved in the downstream market for the buyer that is not contractible. Thus, the side-payment to the suppliers cannot be contingent on the demand realization. We assume $R(q)$ is strictly concave with $R(0) = 0$, $R'(0) = \infty$ and $R'(\infty) = 0$, so that quantity ordered by the buyer is non-zero and bounded. Without this assumption the results in this paper would remain qualitatively the same; however, the optimal mechanism would have a reservation cost above which the buyer will not order anything. Characterizing the optimal reserve cost is straightforward and is well-studied (see Dasgupta and Spulber (1990)).

Supplier i, $i = 1, \ldots, n$, has a constant marginal production cost $c_i \in [\underline{c}, \bar{c}] \subset (0, \infty)$ and finite capacity $q_i \in [\underline{q}, \bar{q}] \subset (0, \infty)$. The joint distribution function of marginal cost c_i and production capacity q_i is denoted by F_i. We assume that (c_i, q_i) and (c_j, q_j) are independently distributed when $i \neq j$, i.e. our model is an *independent private value* (IPV) model. We assume that distribution functions $\{F_i\}_{i=1}^n$ are common knowledge; however, the realization (c_i, q_i) is only known to supplier i. The buyer seeks a revenue maximizing procurement mechanism that ensures that all suppliers participate in the auction.

We employ the direct mechanism approach, i.e. the buyer asks suppliers to directly bid their private information (c_i, q_i). The revelation principle ((see Myerson, 1981; Harris and Townsend, 1981)) implies that for any given mechanism one can construct a direct mechanism that has the same point-wise allocation and transfer payment as the given mechanism. Since both mechanisms result in the same expected profit for the buyer, it follows that there is no loss of generality in restricting oneself to direct mechanisms.

We denote the true type of supplier by $\mathbf{b}_i = (c_i, q_i)$ and the supplier i's bid by $\widehat{\mathbf{b}}_i = (\hat{c}_i, \hat{q}_i)$. Let $\mathbf{b} = (\mathbf{b}_1, ..., \mathbf{b}_n)$ and $\widehat{\mathbf{b}} = (\widehat{\mathbf{b}}_1, ..., \widehat{\mathbf{b}}_n)$. Let $\mathbf{B} \equiv \left([\underline{c}, \bar{c}] \times [\underline{q}, \bar{q}]\right)^n$ denote the type space. A procurement mechanism consists of

1. an allocation function $\mathbf{x} : \mathbf{B} \to \mathbb{R}^n_+$ that for each bid vector $\widehat{\mathbf{b}}$ specifies the quantity to be ordered from each of the suppliers, and
2. a transfer payment function $\mathbf{t} : \mathbf{B} \to \mathbb{R}^n$ that maps each bid vector $\widehat{\mathbf{b}}$ to the transfer payment from the buyer to the suppliers.

The buyer seek an allocation function $\mathbf{x}$ and a transfer function $\mathbf{t}$ that maximizes the ex-ante expected profit

$$\Pi(\mathbf{x}, \mathbf{t}) \equiv \mathbb{E}_{\mathbf{b}} \left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b}) \right) - \sum_{i=1}^{n} t_i(\mathbf{b}) \right]$$

subject to the following constraints.

1. feasibility: $x_i(\mathbf{b}) \le q_i$ for all $i = 1, \ldots, n$, and $\mathbf{b} \in \mathbf{B}$,
2. *incentive compatibility* (**IC**): Conditional on their beliefs about the private information of other bidders, truthfully revealing their private information is weakly dominant for all suppliers, i.e.

$$(c_i, q_i) \in \underset{\substack{\hat{c}_i \in [\underline{c}, \bar{c}] \\ \hat{q}_i \in [\underline{q}, q_i]}}{\operatorname{argmax}} \mathbb{E}_{\mathbf{b}_{-i}} \left\{ t_i((\hat{c}_i, \hat{q}_i), \mathbf{b}_{-i}) - c_i x_i((\hat{c}_i, \hat{q}_i), \mathbf{b}_{-i}) \right\}, \qquad i = 1, \ldots, n, \tag{1}$$

   Note that the range for the capacity bid $\hat{q}_i$ is $[\underline{q}, q_i]$, i.e. we do not allow the supplier to overbid capacity. This can be justified by assuming that the supplier incurs a heavy penalty for not being able to deliver the allocated quantity.

3. *individual rationality* (**IR**): The expected interim surplus of each supplier firm is non-negative, for all $i = 1, \ldots, n$, and $\mathbf{b} \in \mathbf{B}$, i.e.

$$\pi_i(\mathbf{b}_i) \equiv \mathbb{E}_{\mathbf{b}_{-i}} \left[t_i(\mathbf{b}) - c_i x_i(\mathbf{b}) \right] = T_i(c_i, q_i) - c_i X_i(c_i, q_i) \ge 0. \tag{2}$$

   Here we have assumed that the outside option available to the suppliers is constant and is normalized to zero.

In this paper, we use **IC** and **IR** as a shorthand for the incentive compatibility and individual rationality, respectively; and we mean Bayesian incentive compatibility and Bayesian individual rationality, unless specified otherwise.

For any procurement mechanism $(\mathbf{x}, \mathbf{t})$, the *offered* expected surplus $\rho_i(\hat{c}_i, \hat{q}_i)$ when supplier i bids $(\hat{c}_i, \hat{q}_i)$ is defined as follows

$$\rho_i(\hat{c}_i, \hat{q}_i) = T_i(\hat{c}_i, \hat{q}_i) - \hat{c}_i X_i(\hat{c}_i, \hat{q}_i)$$

The offered surplus is simply a convenient way of expressing the expected transfer payment. The expected surplus $\pi_i(c_i, q_i)$ of supplier i with true type (c_i, q_i) when she bids $(\hat{c}_i, \hat{q}_i)$ is given by

$$\pi_i(c_i, q_i) = T_i(\hat{c}_i, \hat{q}_i) - c_i X_i(\hat{c}_i, \hat{q}_i) = \rho_i(\hat{c}_i, \hat{q}_i) + (\hat{c}_i - c_i) X_i(\hat{c}_i, \hat{q}_i).$$

The true surplus π_i equals the offered surplus ρ_i if the mechanism $(\mathbf{x}, \mathbf{t})$ is **IC**.

To further motivate the procurement mechanism design problem, we elaborate on a supplier's incentives to lie about capacity and then consider some illustrative special cases.

2.1 Incentive to Underbid Capacity

In this section we show that auctions that ignore the capacity information are not incentive compatible. In particular, the suppliers have an incentive to underbid capacity.

Suppose we ignore the private capacity information and implement the classic K^{th} price auction where the marginal payment to the supplier is equal to the marginal cost of the first losing supplier, i.e. lowest cost supplier among those that did not receive any allocation. Then truthfully bidding the marginal cost is a dominant strategy. However, we show below that in this mechanism the suppliers have an incentive to underbid capacity. Underbidding creates a fake shortage resulting in an increase in the transfer payment that can often more than compensates the loss due to a possible decrease in the allocation. The following example illustrates these incentives in dominant strategy and Bayesian framework.

Example 1. Consider a procurement auction with three capacitated suppliers implemented as the K^{th} price auction. Let $\underline{c} = 1, \bar{c} = 5, \underline{q} = .01$ and $\bar{q} = 6$. Suppose the capacity realization is $(q_1, q_2, q_3) = (5, 1, 5)$ and the marginal cost realization is $(c_1, c_2, c_3) = (1, 1, 5)$. Suppose the buyer wants to procure 5 units and that the spot price, i.e. the outside publicly known cost at which the buyer can procure unlimited quantity is equal to 10. (We need to have an outside market when modeling fixed quantity auction because the realized total capacity of the suppliers can be less than the fixed quantity that needs to be procured.)

Assume that suppliers 2 and 3 bid truthfully. Consider supplier 1. If she truthfully reveals her capacity, her surplus is \$0; however, if she bids $\hat{q}_1 = 4 - \epsilon$, her surplus is equal to \$$9(4 - \epsilon)$. Thus, bidding truthfully is not a dominant strategy for supplier 1.

Next, we show that for appropriately chosen asymmetric prior distributions supplier 1 has incentives to underbid capacity even in the Bayesian framework. Assume that the marginal cost and capacity are independently distributed. Let $(c_1, q_1) = (1, 5)$. Thus $\mathbb{P}((1 \leq c_2) \cap (1 \leq c_3)) = 1$. Let the capacity distribution F_i^q, $i = 2, 3$, be such that $\mathbb{P}(q_2 + q_3 \leq 1) > 1 - \epsilon$ for some $0 < \epsilon \ll 1$. Then the expected surplus $\pi_1(1, 5)$, if supplier 1 bids her capacity truthfully, is upper bounded by $5 \times (\bar{c} - 1) = 20$. On the other hand the expected surplus if she bids $4 - \epsilon$ is lower bounded by $9 \times (4 - \epsilon) \times (1 - \epsilon)$. Thus, supplier 1 has ex-ante incentive to underbid capacity. ■

Figure 1 shows two uniform price auction mechanisms, the K^{th} price auction and the market clearing mechanism. In our model, the suppliers can change the supply ladder curve both in terms of location of the jumps (by misreporting costs) and the magnitude of the jump (by misreporting capacity). We know that in a model with commonly known capacities, the fixed quantity optimal auction can be implemented as K^{th} price auction. We showed in the example above that in the K^{th} price auction with privately known capacity, the suppliers can "*game*" the mechanism.

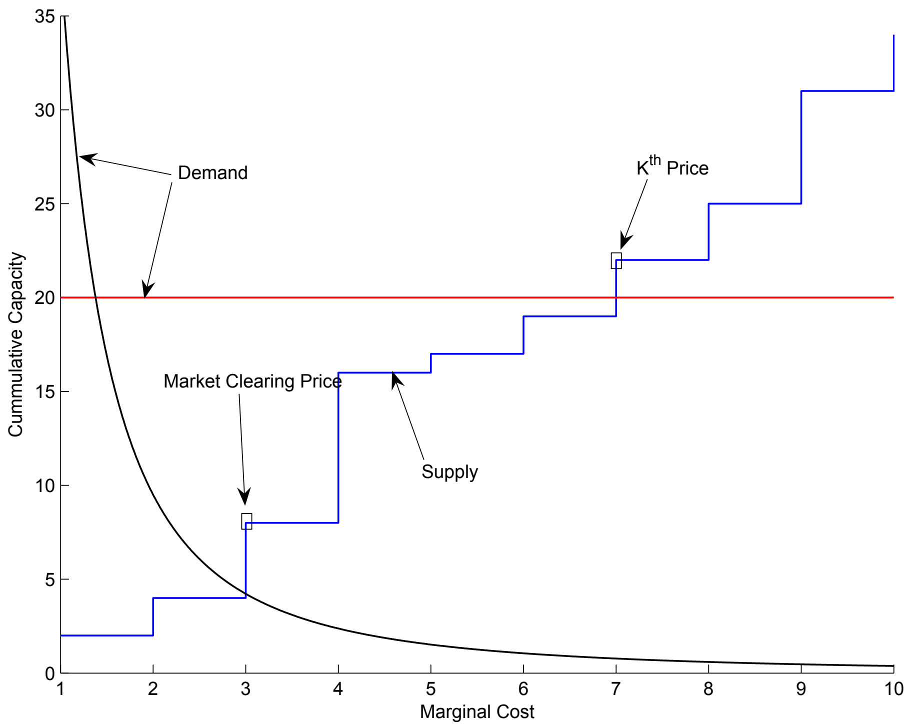

Figure 1: Uniform price auctions: K^{th} price auction and market clearing price auction

This effect is also true if prices are determined by the market clearing condition. Suppose the suppliers truthfully reveal their marginal costs and the buyer aggregates these bids to form the supply curve $Q(p) = \sum_{i=1}^{n} \hat{q}_i \mathbf{1}_{\{c_i \leq p\}}$. The demand curve $D(p)$ in this context is given by

$$D(p) = \underset{u \geq 0}{\operatorname{argmax}} \, [R(u) - pu] = (R')^{-1}(p).$$

Thus, the equilibrium price p^* is given by the solution of the market clearing condition $(R')^{-1}(p) = Q(p^*)$ (see Figure 1). The model primitives ensure that the market clearing price $p^* \in (0, \infty)$. In such a setting, as in the K-th price auction, the supplier with low cost and high capacity can at times increase surplus by underbidding capacity because the increase in the marginal (market clearing) price can offset the decrease in allocation.

The above discussion shows that both the K-th price auction and the market-clearing mechanism are not truth revealing. In § 2.2.3 we show that if the suppliers bid the cost truthfully for exogenous reasons, the buyer can extract all the surplus, i.e. the buyer does not pay any information rent to the suppliers for the capacity information. In this mechanism the transfer payments are simply the true costs of the supplier and the quantity allocated is a monotonically decreasing function of the marginal cost. This optimal mechanism is *discriminatory* and unique. In particular, with privately known capacities, there does not exist a *uniform price* optimal auction. Ausubel (2004) shows that a modified market clearing mechanism, where items are awarded at the price that they are "clinched", is efficient, i.e. socially optimal ((see, also Ausubel and Cramton, 2002)).

2.2 Relaxations

In this section we discuss some special cases of the procurement mechanism design problem formulated in § 2.

2.2.1 Full Information (or First-best) Solution

Suppose all suppliers bid truthfully. It is clear that in this setting the surplus of each supplier would be identically zero. Denote the marginal cost of supplier firm with i^{th} lowest marginal cost by $c_{[i]}$ and it's capacity by $q_{[i]}$. Then the piece-wise convex linear cost function faced by the buyer is given by

$$c(y) = \sum_{j=1}^{i-1} q_{[j]} c_{[i]} + \left(y - \sum_{j=1}^{i-1} q_{[j]} \right) c_{[i]} \quad \text{for } \sum_{j=1}^{i-1} q_{[j]} \leq y \leq \sum_{j=1}^{i} q_{[j]} \tag{3}$$

The optimal procurement strategy for the buyer is the same as that of a buyer facing a single supplier with piece wise linear convex production cost $c(y)$. Clearly, multi-sourcing is optimal with a number of lowest cost suppliers producing at capacity and at most one supplier producing below capacity.

Multiple sourcing can also occur in an uncapacitated model when the production costs are nonlinear. We expect that a risk averse buyer would also find it advantageous to multi-source to diversify the ex-ante risk due to the asymmetric information. Since, to the best of our knowledge, the problem of optimal auctions with a risk averse principal has not been fully explored in the literature, this remains a conjecture.

2.2.2 Second-degree Price Discrimination with a Single Capacitated Supplier

Suppose there is a single supplier with privately known marginal cost and capacity. Suppose the capacity and cost are independently distributed. Let $F(c)$ and $f(c)$ denote, respectively, the cumulative distribution function (CDF) and density of the marginal cost c and suppose the hazard rate $\frac{f(c)}{F(c)}$ is monotonically decreasing, i.e. we are in the so-called regular case (Myerson (1981)). Note that this is a standard adverse selection problem; the procurement counterpart of second degree price discrimination in the monopoly pricing model.

We will first review the optimal mechanism when the supplier is uncapacitated. Using the indirect utility approach, the buyer's problem can formulated as follows.

$$\max_{\substack{x(\cdot)\geq 0 \\ x(\cdot)\text{monotone}}} \mathbb{E}_c \left[R(x(c)) - \left(c + \frac{F(c)}{f(c)} \right) x(c) \right]. \tag{4}$$

Let $x^*(c)$ denote the optimal solution of the relaxation of (4) where one ignores the monotonicity assumption, i.e.

$$x^*(c) \in \operatorname*{argmax}_{x \geq 0} \left\{ R(x) - \left(c + \frac{F(c)}{f(c)} \right) x \right\}.$$

Then, regularity implies that x^* is a monotone function of c, and is, therefore, feasible for (4). The transfer payment $t^*(c)$ that makes the optimal allocation x^* **IC** is given by

$$t^*(c) = cx^*(c) + \int_c^{\bar{c}} x^*(u) du.$$

Since the optimal allocation $x^*(c)$ and the transfer payment $t^*(c)$ are both monotone in c, the cost parameter c can be eliminated to obtain the transfer t directly in terms of the allocation x, i.e. a *tariff* $t^*(x)$. The indirect tariff implementation is very appealing for implementation as it can "posted" and the supplers can simply self-select the production quantity based on the posted tariff.

Now consider the case of a capacitated supplier. Feasibility requires that for all $c \in [\underline{c}, \bar{c}]$, $0 \leq x(c) \leq q$. Suppose the supplier bids the capacity truthfully. (We justify this assumption below.) Then the buyer's problem is given by

$$\max_{\substack{x(\cdot,\cdot)\geq 0 \\ x(\cdot,q)\text{monotone}}} \mathbb{E}_{(c,q)} \left[R(x(c,q)) - \left(c + \frac{F(c)}{f(c)} \right) x(c,q) \right] \tag{5}$$

where F denotes the marginal distribution of the cost. Set the allocation $\hat{x}(c,q) = \min\{x^*(c), q\}$, where x^* denotes the optimal solution of the uncapacitated problem (4). Then $\hat{x}$ is clearly feasible for (5). Moreover,

$$\hat{x}(c,q) \in \operatorname*{argmax}_{0 \leq x \leq q} \left\{ R(x) - \left(c + \frac{F(c)}{f(c)} \right) x \right\}.$$

Thus, $\hat{x}$ is an optimal solution of (5). As before, set transfer payment $\hat{t}(c,q) = c\hat{x}(c,q) + \int_c^{\bar{c}} \hat{x}(u,q)du$. Then, the supplier surplus in the solution $(\hat{x}, \hat{t})$ is non-decreasing in the capacity bid q. Therefore, it is weakly dominant for the supplier to bid the capacity truthfully, and our initial assumption is justified. Note that the supplier surplus $\hat{\pi}(c,q) = \int_c^{\bar{c}} \hat{x}(u,q)du$.

The fact that the capacitated solution $\hat{x}(c,q) = \min\{x^*(c), q\}$ is simply a truncation of the uncapacitated solution $x^*(c)$ allows one to implement it in a very simple manner. Suppose the buyer offers the seller the tariff $t^*(x)$ corresponding to the uncapacitated solution. Then the solution $\tilde{x}$ of the seller's optimization problem $\max_{0 \leq x \leq q}\{t^*(x) - cx\}$ is given by

$$\tilde{x} = \min\{x^*(c), q\} = \hat{x}(c,q),$$

i.e. the quantity supplied is the same as that dictated by the optimal capacitated mechanism. Define $c_q = \sup\{c \in [\underline{c}, \bar{c}] : x^*(c) \geq q\}$. Then the monotonicity of $x^*(c)$ implies that

$$\tilde{x} = \hat{x}(c,q) = \begin{cases} x^*(c), & c > c_q, \\ q, & c \leq c_q. \end{cases}$$

Then, for all $c > c_q$, the supplier requests $x^*(c)$ and receives a surplus

$$\begin{aligned} \tilde{\pi}(c) = t^*(x^*(c)) - cx^*(c) = \int_c^{\bar{c}} x^*(u)du \\ = \int_c^{\bar{c}} \min\{x^*(u), q\}du = \int_c^{\bar{c}} \hat{x}(u,q)du = \hat{\pi}(c,q). \end{aligned}$$

For $c \leq c_q$, the supplier request q and the surplus

$$\begin{aligned} \tilde{\pi}(c) &= t^*(q) - cq, \\ &= t^*(x^*(c_q)) - c_q q + (c_q - c)q, \\ &= \pi^*(c_q) + (c_q - c)q = \int_{c_q}^{\bar{c}} x^*(u)du + \int_c^{c_q} q du = \int_c^{\bar{c}} \hat{x}(u,q)du = \hat{\pi}(c,q). \end{aligned}$$

Thus, the supplier surplus in the tariff implementation is $\hat{\pi}(c,q)$, the surplus associated with optimal capacitated mechanism. Consequently, it follows that the "full" tariff implements the capacitated optimal mechanism! This immediately implies that the buyer does need to know the capacity of the supplier, and pays zero information rent for the capacity information. In the next section we show that the assumption of independence of capacity and cost is critical for this result.

2.2.3 Marginal Cost Common Knowledge

Suppose the marginal costs are common knowledge and only the production capacities are privately known. Then the optimal procurement mechanism maximizes

$$\max_{(\mathbf{x},\mathbf{t})} \mathbb{E}_q \left[R\Big(q_i \sum_{i=1}^{n} x_i(q) \Big) - \sum_{i=1}^{n} t_i(q) \right]$$

subject to the constraint that the expected supplier i's surplus $T_i(q_i) - c_i X_i(q_i)$ is weakly increasing in q_i (**IC**) and nonnegative (**IR**) for all suppliers i.

Not surprisingly, the first-best or the full-information solution works in this case. Set the transfer payment equal to the production costs of the supplier, i.e. $t_i(q) = c_i x_i(q)$. Then the supplier surplus is zero and the buyer's optimization problem reduces to the full-information case. Since the full-information allocation $x_i(\hat{q}_i, q_{-i})$ is weakly increasing in $\hat{q}_i$ for all q_{-i}, bidding the true capacity is a weakly dominant strategy for the suppliers. Thus, the buyer can effectively ignore the **IC** constraints above and follow the full information allocation scheme and extract all the supplier surplus. The fact that, conditional on knowing the cost, the buyer does not offer any informational rent for the capacity information is crucial to the result in the next section.

3 Characterizing Optimal Direct Mechanism

We use the standard indirect utility approach to characterize all incentive compatible and individually rational direct mechanisms and the minimal transfer payment function that implements a given incentive compatible allocation rule (Lemma 1). The characterization of the transfer payment allows us to write the expected profit of the buyer for a given incentive compatible allocation rule as a function of the allocation rule and the offered surplus $\rho_i(\bar{c}, q)$ (Theorem 1). To proceed further, we make the following assumption.

Assumption 1. *For all $i = 1, 2, \cdots, n$, the joint density $f_i(c_i, q_i)$ has full support.*

Note that if Assumption 1 holds then the conditional density $f_i(c_i|q_i)$ also has full support.

Lemma 1. *Procurement mechanisms with capacitated suppliers satisfy the following.*

1. *A feasible allocation rule $\mathbf{x} : \mathbf{B} \to \mathbb{R}^n_+$ is* **IC** *if, and only if, the expected allocation $X_i(c_i, q_i)$ is non-increasing in the cost parameter c_i for all suppliers $i = 1, \ldots, n$.*

2. *A mechanism $(\mathbf{x}, \mathbf{t})$ is* **IC** *and* **IR** *if, and only if, the allocation rule $\mathbf{x}$ satisfies (a) and the offered surplus $\rho_i(\hat{c}_i, \hat{q}_i)$ when supplier i bids $(\hat{c}_i, \hat{q}_i)$ is of the form*

$$\rho_i(\hat{c}_i, \hat{q}_i) = \rho_i(\bar{c}, \hat{q}_i) + \int_{\hat{c}_i}^{\bar{c}} X_i(u, \hat{q}_i) du \tag{6}$$

with $\rho_i(\hat{c}_i, \hat{q}_i)$ *non-negative and non-decreasing in* $\hat{q}_i$ *for all* $\hat{c}_i \in [\underline{c}, \bar{c}]$ *and* i*.*

Remark 1. *Recall that the offered surplus* ρ_i *is, in fact, equal to the surplus* π_i *when the allocation rule* $\mathbf{x}$ *(and the associated transfer payment* $\mathbf{t}$*) is* **IC**.

Proof: Fix the mechanism $(\mathbf{x}, \mathbf{t})$. Then the supplier i expected surplus $\pi_i(c_i, q_i)$ is given by

$$\pi_i(c_i, q_i) = \max_{\substack{\hat{c}_i \in [\underline{c}, \bar{c}] \\ \hat{q}_i \in [\underline{q}, q_i]}} \{T_i(\hat{c}_i, \hat{q}_i) - c_i X_i(\hat{c}_i, \hat{q}_i)\}. \tag{7}$$

Note that the capacity bid $\hat{q}_i \leq q_i$, the true capacity. This plays an important role in the proof. From (7), it follows that for all fixed $q \in [\underline{q}, \bar{q}]$, the surplus $\pi_i(c_i, q_i)$ is convex in the cost parameter c_i. (There is, however, no guarantee that $\pi_i(c_i, q_i)$ is jointly convex in (c_i, q_i).) Consequently, for all fixed $q \in [\underline{q}, \bar{q}]$, the function $\pi_i(c_i, q_i)$ is absolutely continuous in c and differentiable almost everywhere in c.

Since $\mathbf{x}$ is **IC**, it follows that (c_i, q_i) achieves the maximum in (7). Thus, in particular,

$$c_i \in \operatorname*{argmax}_{\hat{c}_i \in [\underline{c}, \bar{c}]} \{T_i(\hat{c}_i, q_i) - c_i X_i(\hat{c}_i, q_i)\}, \tag{8}$$

i.e. if supplier i bids capacity q truthfully, it is still optimal for her to bid the cost truthfully. Since $\pi_i(c_i, q_i)$ is convex in c_i, (8) implies that

$$\frac{\partial \pi_i(c, q)}{\partial c} = -X_i(c, q), \quad \text{a.e.} \tag{9}$$

Consequently, $X_i(c, q)$ is non-increasing in c for all $q \in [\underline{q}, \bar{q}]$. This proves the forward direction of the assertion in part (a).

To prove the converse of part (a), suppose $X_i(c_i, q_i)$ is non-increasing in c_i for all q_i. Set the offered surplus

$$\rho_i(\hat{c}_i, \hat{q}_i) = \bar{\rho}_i(\hat{q}_i) + \int_c^{\bar{c}} X_i(u, \hat{q}_i) du$$

where the function $\bar{\rho}_i(\hat{q}_i) \triangleq \rho(\bar{c}, \hat{q}_i)$ is such that $\rho_i(\hat{c}_i, \hat{q}_i)$ is non-decreasing in $\hat{q}_i$ for all $\hat{c}_i \in [\underline{c}, \bar{c}]$. There are many feasible choices for $\bar{\rho}(\hat{q}_i)$. In particular, if $\frac{\partial X_i(c,q)}{\partial q}$ exists a.e., one can set,

$$\bar{\rho}_i(\hat{q}_i) = \sup_{c_i \in [\underline{c}, \bar{c}]} \Big\{ \int_{\underline{q}}^{q_i} \int_{c_i}^{\bar{c}} \Big(\frac{\partial X_i(t, z)}{\partial z} \Big)^{-} dt dz \Big\}.$$

For any such choice of $\bar{\rho}_i$, the supplier i surplus

$$
\begin{aligned}
\pi_i(\hat{c}_i, \hat{q}_i) &= \rho_i(\hat{c}_i, \hat{q}_i) + (\hat{c}_i - c_i) X_i(\hat{c}_i, \hat{q}_i), \\
&= \bar{\rho}_i(\hat{q}_i) + \int_{\hat{c}_i}^{\bar{c}} X_i(u, \hat{q}_i) du + (\hat{c}_i - c_i) X_i(\hat{c}_i, \hat{q}_i), \\
&= \bar{\rho}_i(\hat{q}_i) + \int_{c_i}^{\bar{c}_i} X_i(u, \hat{q}_i) du + \int_{\hat{c}_i}^{c_i} X_i(u, \hat{q}_i) du + (\hat{c}_i - c_i) X_i(\hat{c}_i, \hat{q}_i), \\
&\leq \bar{\rho}_i(\hat{q}_i) + \int_{c_i}^{\bar{c}} X_i(u, \hat{q}_i) du, \qquad (10) \\
&\leq \bar{\rho}_i(q_i) + \int_{c_i}^{\bar{c}} X_i(u, q_i) du, \qquad (11) \\
&= T_i(c_i, q_i) - c_i X_i(c_i, q_i) = \pi_i(c_i, q_i),
\end{aligned}
$$

where (10) follows from the fact that $X_i(c, q)$ in non-increasing in c for all fixed q and (11) follows from the $\rho_i(\hat{c}_i, \hat{q}_i)$ is non-decreasing in $\hat{q}_i$ and $\hat{q}_i \leq q_i$. Thus, we have established that it is weakly dominant for supplier i to bid truthfully, or equivalently $\mathbf{x}$ is an incentive compatible allocation. From (9) we have that whenever $\mathbf{x}$ is **IC** we must have that the supplier surplus is of the form

$$
\pi_i(c_i, q_i) = \pi_i(\bar{c}, q_i) + \int_{c}^{\bar{c}} X_i(u, q_i) du.
$$

Since $\mathbf{x}$ is **IR**, $\pi_i(\bar{c}_i, q_i) \geq 0$, and, since $\mathbf{x}$ is **IC**,

$$
q_i \in \underset{\hat{q}_i \leq q_i}{\operatorname{argmax}} \left\{ T_i(c_i, \hat{q}_i) - c_i X_i(c_i, \hat{q}_i) \right\} = \underset{\hat{q}_i \leq q_i}{\operatorname{argmax}} \left\{ \pi_i(c_i, \hat{q}_i) \right\}.
$$

Thus, we must have that $\pi_i(c_i, q_i)$ is non-decreasing in q_i for all $c_i \in [\underline{c}, \bar{c}]$. This establishes the forward direction of part (b).

Suppose the offered surplus if of the form (6) then $(\mathbf{x}, \mathbf{t})$ satisfies **IR**. Since $X_i(c_i, q_i)$ is non-increasing in c_i for all q_i, it follows that $\pi_i(c_i, q_i)$ is convex in c_i for all q_i and $\frac{\partial \pi_i(c_i, q_i)}{\partial c_i} = -X_i(c_i, q_i)$. Consequently,

$$
\pi_i(\hat{c}_i, \hat{q}_i) = \rho_i(\hat{c}_i, \hat{q}_i) + (c_i - \hat{c}_i)\big(-X_i(\hat{c}_i, \hat{q}_i)\big) \leq \pi_i(c_i, \hat{q}_i) \leq \pi_i(c_i, q_i),
$$

where the last inequality follows from the fact that $\pi_i(c_i, q_i)$ is non-decreasing in q_i for all c_i and $\hat{q}_i \leq q_i$. Thus, we have establishes that $(\mathbf{x}, \mathbf{t})$ is **IC**. ∎

Next, we use the results in Lemma 1 to characterize the buyer's expected profit.

Theorem 1. *Suppose Assumption 1 holds. Then the buyer profit* $\Pi(\mathbf{x}, \mathbf{t})$ *corresponding to any* *feasible allocation rule* $\mathbf{x} : \mathbf{B} \rightarrow \mathbb{R}_+^n$ *that satisfies* **IC** *and* **IR** *is given by*

$$\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}}) = \mathbb{E}_b \left[R\left(\sum_{i=1}^{n} x_i(b)\right) - \sum_{i=1}^{n} x_i(b) H_i(c_i, q_i) - \sum_{i=1}^{n} \bar{\rho}_i(q_i) \right], \tag{12}$$

where $\bar{\rho}_i(q_i)$ *is the surplus offered when the supplier* i *bid is* $(\bar{c}, q_i)$ *and* $H_i(c_i, q_i)$ *denotes the virtual costdefined to be* $H_i(c_i, q_i) \equiv c_i + \frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}$.

Remark 2. *Theorem 1 implies that the buyer's profit is determined by* both *the allocation rule* $\mathbf{x}$ *and offered surplus* $\bar{\boldsymbol{\rho}}(q)$ *when supplier* i *bid is* $(\bar{c}, q)$*. We emphasize this by denoting the buyer profit by* $\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}})$ *.*

Proof: From Lemma 1, we have that the offered supplier i surplus $\rho_i(c_i, q_i)$ under any **IC** and **IR** allocation rule $\mathbf{x}$ is of the form

$$\rho_i(c_i, q_i) = \rho_i(\bar{c}, q_i) + \int_{c_i}^{\bar{c}} X_i(t, q_i) dt$$

Thus, the buyer profit function is

$$\begin{aligned} \Pi =& \mathbb{E}_{\mathbf{b}} \left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b})\right) - \sum_{i=1}^{n} \left(c_i x_i(\mathbf{b}) + \rho_i(\bar{c}, q_i)\right) \right] \\ &- \sum_{i=1}^{n} \left(\int_{\underline{q}}^{\bar{q}} \int_{\underline{c}}^{\bar{c}} \int_{c_i}^{\bar{c}} X_i(u_i, q_i) du_i f_i(c_i, q_i) dc_i dq_i \right). \end{aligned}$$

By interchanging the order of integration, we have

$$\begin{aligned} \int_{\underline{c}}^{\bar{c}} dc_i f_i(c_i, q_i) \int_{c_i}^{\bar{c}} du_i X_i(u_i, q_i) &= \int_{\underline{c}}^{\bar{c}} du_i X_i(u_i, q_i) \int_{\underline{c}}^{t} dc f_i(c, q_i) \\ &= \int_{\underline{c}}^{\bar{c}} X_i(c_i, q_i) F_i(c_i \mid q_i) f_i(q_i) dc_i. \end{aligned}$$

Substituting this back into the expression for profit, we get

$$\begin{aligned}\Pi(\mathbf{x}) &= \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b})\right) - \sum_{i=1}^{n}\left(c_i x_i(\mathbf{b}) + \rho_i(\bar{c}, q_i)\right)\right] \\ &\quad - \sum_{i=1}^{n}\left(\int_{\underline{q}}^{\bar{q}}\int_{\underline{c}}^{\bar{c}} X_i(c_i, q_i) F_i(c_i|q_i) f_i(q_i) dc_i dq_i\right), \\ &= \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b})\right) - \sum_{i=1}^{n}\left(c_i x_i(\mathbf{b}) + \rho_i(\bar{c}, q_i)\right)\right] - \sum_{i=1}^{n}\mathbb{E}_{\mathbf{b}}\left[x_i(\mathbf{b})\frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}\right] \\ &= \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^{n} x_i(\mathbf{b})\right) - \sum_{i=1}^{n}\left(c_i + \frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}\right) x_i(\mathbf{b}) - \sum_{i=1}^{n}\rho_i(\bar{c}, q_i)\right].\end{aligned}$$

This establishes the result. ■

The virtual marginal costs $H_i(c, q)$ in our model are very similar to the virtual marginal costs in the uncapacitated reverse auction model; except that the virtual costs are now defined in terms of the distribution of the marginal cost c_i *conditioned* on the capacity bid q_i. Thus, the capacity bid provides information only if the cost and capacity are correlated. Next, we characterize the optimal allocation rule under the regularity Assumption 2 and to a limited extent under general model primitives.

3.1 Optimal Mechanism in the Regular Case

In this section, we make the following additional regularity assumption about the monotonicity of the virtual marginal costs.

Assumption 2 (Regularity). *For all $i = 1, 2, \cdots, n$, the virtual cost function*

$$H_i(c_i, q_i) \equiv c_i + \frac{F_i\,(c_i|q_i)}{f_i(c_i|q_i)}$$

is non-decreasing in c_i and non-increasing in q_i.

Assumption 2 is called the *regularity* condition. This regularity condition on virtual cost is commonly assumed in literature on procurement auctions, except that we require monotonicity in both the cost variable as well as the capacity variable. It is satisfied when the conditional density of the marginal cost given capacity is log concave in c_i, and the production cost and capacity are, loosely speaking, "negatively affiliated" in such a way that $\frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}$ is non-increasing in q_i. This is true, for example, when the cost and capacity are independent.

For $\mathbf{b} \in \mathbf{B}$, define

$$\mathbf{x}^*(\mathbf{b}) \equiv \operatorname*{argmax}_{0 \le \mathbf{x} \le \mathbf{q}} \left\{R\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} x_i H_i(c_i, q_i)\right\}, \tag{13}$$

where the inequality $0 \leq \mathbf{x} \leq \mathbf{q}$ is interpreted component-wise. We call $\mathbf{x}^* : \mathbf{B} \to \mathbb{R}^n_+$ the point-wise optimal allocation rule. Since (13) is identical to the full information problem with the cost c_i replaced by the *virtual cost* $H_i(c_i, q_i)$, it follows that (13) can be solved by aggregating all the suppliers into one meta-supplier. Denote the virtual cost of supplier with i^{th} lowest virtual cost by $h_{[i]}$ and the corresponding capacity by $q_{[i]}$. Then the buyer faces a piece-wise convex linear cost function $h(q)$ given by

$$h(q) = \sum_{j=1}^{i-1} q_{[j]} h_{[i]} + \left(q - \sum_{j=1}^{i-1} q_{[j]}\right) c_{[i]}, \tag{14}$$

for $\sum_{j=1}^{i-1} q_{[j]} \leq q \leq \sum_{j=1}^{i} q_{[j]}$, $i = 1, \ldots, n$, where $\sum_{j=1}^{0} q_{[j]}$ is set to zero. From the structure of the supply curve it follows that the optimal solution of (13) is of the form

$$x^*_{[i]} \equiv \begin{cases} q_{[i]}, & [i] < [i]^*, \\ \leq q_{[i]}, & [i] = [i]^*, \\ 0 & \text{otherwise}, \end{cases} \tag{15}$$

where $1 \leq [i]^* \leq n$.

Lemma 2. *Suppose Assumption 2 holds. Let $\mathbf{x}^* : \mathbf{B} \to \mathbb{R}^n_+$ denote the point-wise optimal defined in (13).*

1. *$x^*_i((c_i, q_i), \mathbf{b}_{-i})$ is non-increasing in c_i for all fixed q_i and $\mathbf{b}_{-i}$. Consequently, $X_i(c_i, q_i)$ is non-increasing in c_i for all q_i.*

2. *$x^*_i((c_i, q_i), \mathbf{b}_{-i})$ is non-decreasing in q_i for all fixed c_i and $\mathbf{b}_{-i}$. Therefore, $X_i(c_i, q_i)$ is non-decreasing in q_i for all fixed c_i.*

Proof: From (15) it is clear that $\mathbf{x}^*((c_i, q_i), \mathbf{b}_{-i})$ is non-increasing in the virtual cost $H_i(c_i, q_i)$. When Assumption 2 holds, the virtual cost $H_i(c_i, q_i)$ is non-decreasing in c_i for fixed q_i; consequently, the allocation x^*_i is non-increasing in the capacity bid q_i for fixed c_i and $\mathbf{b}_{-i}$. Part (a) is established by taking expectations of $\mathbf{b}_{-i}$. A similar argument proves (b). ■

We are now in position to prove the main result of this section.

Theorem 2. *Suppose Assumption 1 and 2 hold. Let $\mathbf{x}^*$ denote the point-wise optimal solution defined in (13). For $i = 1, \ldots, n$, set the transfer payment*

$$t^*_i(\widehat{\mathbf{b}}) = \hat{c}_i X^*_i(c_i, q_i) + \int_{\hat{c}_i}^{\bar{c}} X^*_i((u, \hat{q}_i))du. \tag{16}$$

Then $(\mathbf{x}^*, \mathbf{t}^*)$ is Bayesian incentive compatible revenue maximizing procurement mechanism.

Proof: From (12), it follows that the buyer profit

$$\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}}) \le \mathbb{E}_{\mathbf{b}} \left[\max_{\mathbf{0} \le \mathbf{x} \le \mathbf{q}} \left\{ R\left(\sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} x_i H_i(c_i, q_i) \right\} \right] = \Pi(\mathbf{x}^*, \mathbf{0}).$$

Thus, all that remains to be shown is that the offered surplus ρ_i^* corresponding to the transfer payment $\mathbf{t}^*$ satisfies $\bar{\rho}_i^*(q_i) = \rho_i^*(\bar{c}_i, q_i) \equiv 0$, and $(\mathbf{x}^*, \mathbf{t}^*)$ is **IC** and **IR**.
From (16), it follows that the offered surplus

$$\rho_i^*(\hat{c}_i, \hat{q}_i) = \int_{\hat{c}_i}^{\bar{c}} X_i^*(u, \hat{q}_i) du. \tag{17}$$

Thus, $\bar{\rho}_i^*(q_i) = \rho_i^*(\bar{c}, q_i) \equiv 0$.

Next, Lemma 2 (a) implies that $X_i^*((\hat{c}_i, \hat{q}_i), \mathbf{b}_{-i})$ is non-increasing in c_i for all q_i. From Lemma 2 (b), we have that $X_i(u, \hat{q}_i)$ is non-decreasing in $\hat{q}_i$. From (17), it follows that $\pi_i(c_i, q_i)$ is non-decreasing in q_i for all c_i. Now, Lemma 1 (b) allows us to conclude that $(\mathbf{x}^*, \mathbf{t}^*)$ is **IC**.

Since $(\mathbf{x}^*, \mathbf{t}^*)$ satisfies **IC**, the offered surplus $\rho_i^*(c_i, q_i)$ is, indeed, the supplier surplus. Then (17) implies that $(\mathbf{x}^*, \mathbf{t}^*)$ is **IR**. ■

Next, we illustrate the optimal reverse auction on a simple example.

Example 2. Consider a procurement auction with two identical suppliers. Suppose the marginal cost c_i and capacity q_i of each of the suppliers are uniformly distributed over the unit square,

$$f_i(c_i, q_i) = 1 \quad \forall (c_i, q_i) \in [0, 1]^2, i = 1, 2.$$

Therefore, the virtual costs

$$H_i(c_i, q_i) = c_i + \frac{F_i(c_i|q_i)}{f_i(c_i|q_i)} = c_i + c_i = 2c_i \quad \forall c_i \in [0, 1], i = 1, 2.$$

It is clear that this example satisfies Assumption 1 and Assumption 2.

Suppose the buyer revenue function $R(q) = 4\sqrt{q}$. Then, it follows that buyer's optimization problem reduces to the point-wise problem

$$\mathbf{x}^*(\mathbf{c}, \mathbf{q}) = \operatorname*{argmax}_{\mathbf{x} \le \mathbf{q}} \left\{ 4\sqrt{\sum_{i=1}^{2} x_i} - 2\sum_{i=1}^{2} c_i x_i \right\}.$$

The above constrained problem can be easily solved using the Karush-Kuhn-Tucker (KKT) conditions which are sufficient because of strict concavity of the buyer's profit function. For $i = 1, 2$, the solution is given by,

$$x_i^*(c,q) = \begin{cases} \frac{1}{c_i^2} & c_i \le c_{-i},\ q_i \ge \frac{1}{c_i^2}, \\ q_i & c_i \le c_{-i},\ q_i < \frac{1}{c_i^2}, \\ 0 & c_i \ge c_{-i},\ q_{-i} \ge \frac{1}{c_{-i}^2}, \\ \min\left\{\max\left\{0, \frac{1}{c_i^2} - q_{-i}\right\}, q_i\right\} & \text{otherwise.} \end{cases}$$

where $-i$, is the index of the supplier competing with supplier i. The corresponding expected transfer payments are given by equation (16). ■

In order for an allocation rule $\mathbf{x}$ to be Bayesian incentive compatible it is sufficient that the expected allocation $X_i(c_i, q_i)$ be weakly monotone in c_i and q_i. Assumption 2 ensures that the point-wise optimal allocation x_i^* is weakly monotone in c_i and q_i. This stronger property of $\mathbf{x}^*$ can be exploited to show that $\mathbf{x}^*$ can be implemented in the dominant strategy solution concept, i.e. there exist a transfer payment function under which truth telling forms an dominant strategy equilibrium.

Theorem 3. *Suppose Assumption 1 and Assumption 2 hold. For $i = 1, \ldots, n$, let the transfer payment be*

$$t_i^{**}(\widehat{\mathbf{b}}) = \hat{c}_i x_i^*(\widehat{\mathbf{b}}) + \int_{\hat{c}_i}^{\bar{c}} x_i^*((u, \hat{q}_i), \widehat{\mathbf{b}}_{-i}) du. \tag{18}$$

Then, $(\mathbf{x}^*, \mathbf{t}^{**})$ is an dominant strategy incentive compatible, individually rational and revenue maximizing procurement mechanism.

Proof: It is clear that the buyer profit under any dominant strategy **IC** and **IR** mechanism is upper bounded by the profit $\Pi(\mathbf{x}^*, \mathbf{0})$ of the point-wise optimal allocation $\mathbf{x}^*$. From (18), it follows that $(\mathbf{x}^*, \mathbf{t}^{**})$ is ex-post (pointwise) **IR**.

Thus, all that remains is to show that $(\mathbf{x}^*, \mathbf{t}^{**})$ is dominant strategy **IC**. Suppose supplier i bids $(\hat{c}_i, \hat{q}_i)$. Then, for all possible misreports $\hat{\mathbf{b}}$ of suppliers other than i, we have

$$\begin{aligned} & t_i^{**}((\hat{c}_i, \hat{q}_i), \widehat{\mathbf{b}}_{-i}) - \hat{c}_i x_i^*((\hat{c}_i, \hat{q}_i), \widehat{\mathbf{b}}_{-i}) \\ = & \int_{c_i}^{\bar{c}} x_i^*((u, \hat{q}_i), \widehat{\mathbf{b}}_{-i}) du \\ & + \int_{\hat{c}_i}^{c_i} x_i^*((u, \hat{q}_i), \widehat{\mathbf{b}}_{-i}) du - (c_i - \hat{c}_i) x_i^*((\hat{c}_i, \hat{q}_i), \widehat{\mathbf{b}}_{-i}), \\ \le & \int_{c_i}^{\bar{c}} x_i^*((u, \hat{q}_i), \widehat{\mathbf{b}}_{-i}) du,, \qquad (19) \\ \le & \int_{c_i}^{\bar{c}} x_i^*((u, q_i), \widehat{\mathbf{b}}_{-i}) du,, \qquad (20) \\ = & t_i^{**}(\mathbf{b}_i, \widehat{\mathbf{b}}_{-i}) - c_i x_i^*(\mathbf{b}_i, \widehat{\mathbf{b}}_{-i}), \end{aligned}$$

where inequality (19) follows from the fact that $x_i^*((c_i, q_i), \mathbf{b}_{-i})$ is non-increasing in c_i for all $(q_i, \mathbf{b}_{-i})$ (see Lemma 2 (a)) and inequality (20) is a consequence of the fact that $x_i^*((c_i, q_i), \mathbf{b}_{-i})$ is non-decreasing in q_i for all $(c_i, \mathbf{b}_{-i})$ (see Lemma 2 (b)). Thus, truth-telling forms a dominant strategy equilibrium. ■

3.2 Optimal Mechanism in the General Case

In this section, we consider the case when Assumption 2 does not hold, i.e. the distribution of the cost and capacity does not satisfy regularity.

The optimal allocation rule is given by the solution to following optimal control problem

$$\begin{aligned}
\max_{\mathbf{x}(\mathbf{b}),\bar{\boldsymbol{\rho}}(\mathbf{q})} \quad & \mathbb{E}_{\mathbf{b}}\left[R\Big(\sum_{i=1}^{n} x_i(\mathbf{b})\Big) - \sum_{i=1}^{n} H_i(c_i, q_i)x_i(\mathbf{b}) + \bar{\rho}_i(q_i)\right] \\
s.t \quad & 0 \leq x_i(c_i, q_i) \leq q_i \quad \forall i, q_i, c_i \\
& \hat{c}_i \geq c_i \Rightarrow X_i(\hat{c}_i, q_i) \leq X_i(c_i, q_i) \quad \forall q_i, c_i, \hat{c}_i, i \\
& \hat{q}_i \geq q_i \Rightarrow \int_{c_i}^{\bar{c}} (X_i(z, q_i) - X_i(z, \hat{q}_i))dz \leq \bar{\rho}_i(\hat{q}_i) - \bar{\rho}_i(q_i) \quad \forall c_i, q_i, \hat{q}_i, i \\
& 0 \leq \bar{\rho}_i(q_i) \quad \forall q_i, i
\end{aligned} \tag{21}$$

This problem is a very large scale stochastic program and is, typically, very hard to solve numerically. We characterize the solution, under a condition weaker than regularity, which we call *semi-regularity*.

We adapt the standard one dimensional ironing procedure ((see, e.g. Myerson, 1981)) to our problem which has a two-dimensional type space. Let $L(c_i, q_i)$ denote the cumulative density along the cost dimension, i.e.

$$L_i(c_i, q_i) = \int_{\underline{c}}^{c_i} f_i(u, q_i)du$$

Since the density $f_i(c_i, q_i)$ is assumed to be strictly positive, $L_i(c_i, q_i)$ is increasing in c_i, and hence, invertible in the c_i coordinate. Let

$$K_i(p_i, q_i) = \int_{\underline{c}}^{c_i} H_i(u, q_i)f_i(u, q_i)dt$$

where $c_i = L_i(\cdot, q_i)^{-1}(p_i)$. Let $\hat{K}_i$ denote the convex envelop of K_i along p_i, i.e.

$$\hat{K}_i(p_i, q_i) = \inf\Big\{\lambda K_i(a, q_i) + (1-\lambda)K_i(b, q_i) | a, b \in [0, L_i(\bar{c}, q_i)], \\ \lambda \in [0,1], \lambda a + (1-\lambda)b = p_i\Big\}.$$

Define ironed-out virtual cost function $\widehat{H}_i(c_i, q_i)$ by setting it to

$$\widehat{H}_i(c_i, q_i) = \frac{\partial \hat{K}_i}{\partial p}(p, q)\Big|_{p_i = L_i(c_i, q_i), q_i}$$

wherever the partial derivative is defined and extending it to $[\underline{c}, \bar{c}]$ by right continuity.

Lemma 3. *The function K_i, the convex envelop $\hat{K}_i$ and the ironed-out virtual costs $\widehat{H}(c_i, q_i)$ satisfy the following properties.*

1. *$\widehat{H}_i(c_i, q_i)$ is continuous and nondecreasing in c_i for all fixed q_i.*
2. *$\hat{K}_i(0, q_i) = K_i(0, q_i)$, $\hat{K}_i(L_i(\bar{c}, q_i), q_i) = K_i(L_i(\bar{c}, q_i), q_i)$,*
3. *For all q_i and p_i, $\hat{K}_i(p_i, q_i) \leq K_i(p_i, q_i)$.*
4. *Whenever $\hat{K}_i(p_i, q_i) < K_i(p_i, q_i)$, there is an interval (a_i, b_i) containing p_i such that $\frac{\partial}{\partial p}\hat{K}(p, q_i) = c$, a constant, for all $p \in (a_i, b_i)$. Thus, $\widehat{H}_i(c_i, q_i)$ is constant with $c_i \in L_i(\cdot, q_i)^{-1}((a_i, b_i))$.*

See Rockafeller (1970) for the proofs of these assertions. Now, we are ready to state our weaker regularity assumption.

Assumption 3 (Semi-Regularity)**.** *For all $i = 1, 2, \cdots, n$, the ironed out virtual marginal production cost, $\widehat{H}_i(c_i, q_i)$ is non-increasing in q_i.*

From Lemma 3 (a) above, it follows that the semi-regularity implies the usual regularity of $\widehat{H}_i$, i.e. $\widehat{H}_i$ satisfies Assumption 2. Theorem 4 shows that if we use this ironed out virtual cost function in the buyer's profit function instead of the original virtual cost and then pointwise maximize to find the optimal allocation relaxing the monotonicity constraints on the optimal allocation and the side payments $\bar{\rho}_i$, then the resulting mechanism is incentive compatible with $\bar{\rho}_i = 0$ and revenue maximizing.

Theorem 4. *Suppose Assumption 3 holds. Let $\mathbf{x}^{\mathrm{I}} : \mathbf{B} \to \mathbb{R}^n_+$ denote any solution of the pointwise optimization problem*

$$\max_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{q}} \left\{ R\Big(\sum_{i=1}^n x_i\Big) - \sum_{i=1}^n x_i \widehat{H}_i(c_i, q_i) \right\}.$$

Set the transfer payment function

$$t_i^{\mathrm{I}}(\mathbf{b}) = c_i x_i^{\mathrm{I}}(\mathbf{b}) + \int_{c_i}^{\bar{c}} x_i^{\mathrm{I}}((u, q_i), \mathbf{b}_{-i}) du. \tag{22}$$

Then $(\mathbf{x}^{\mathrm{I}}, \mathbf{t}^{\mathrm{I}})$ *is a revenue maximizing, dominant strategy incentive compatible and individually rational procurement mechanism.*

Proof: Let $\mathbf{x}$ be any **IC** allocation and let $\boldsymbol{\rho}$ denote the corresponding offered surplus. Define

$$\widehat{\Pi}(\mathbf{x}, \bar{\boldsymbol{\rho}}) \equiv \mathbb{E}_b \left[R\left(\sum_{i=1}^n x_i(b) \right) - \sum_{i=1}^n x_i(b) \widehat{H}_i(c_i, q_i) - \sum_{i=1}^n \bar{\rho}_i(q_i) \right],$$

i.e. $\widehat{\Pi}(\mathbf{x}, \bar{\boldsymbol{\rho}})$ denotes buyer profit when the virtual costs $H_i(c_i, q_i)$ are replaced by the ironed-out virtual costs $\widehat{H}_i(c_i, q_i)$. Then

$$\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}}) - \widehat{\Pi}(\mathbf{x}, \bar{\boldsymbol{\rho}}) = \int_{\underline{q}}^{\bar{q}} \left[\int_{\underline{c}}^{\bar{c}} \left(\widehat{H}_i(c_i, q_i) - H_i(c_i, q_i) \right) X_i(c_i, q_i) f_i(c_i, q_i) dc_i \right] dq_i$$

The inner integral

$$\begin{aligned}
&\int_{\underline{c}}^{\bar{c}} \left(\widehat{H}_i(c_i, q_i) - H_i(c_i, q_i) \right) X_i(c_i, q_i) f_i(c_i, q_i) dc_i \\
&= \left(\hat{K}_i(c_i, t) - K_i(c_i, t) \right) \Big|_0^{L_i(c_i, q_i)} - \int_{\underline{c}}^{\bar{c}} \left(\hat{K}_i(c_i, q_i) - K_i(c_i, q_i) \right) f_i(c_i, q_i) d_{c_i} \left[X_i(c_i, q_i) \right], \\
&= -\int_{\underline{c}}^{\bar{c}} \left(\hat{K}_i(L(c_i, q_i), q_i) - K_i(L(c_i, q_i), q_i) \right) f_i(c_i, q_i) \partial_{c_i} X_i(c_i, q_i), \qquad (23) \\
&\leq 0, \qquad (24)
\end{aligned}$$

where (23) follows from Lemma 3 (b), and (24) follows from Lemma 3 (c) and the fact that $\partial_{c_i} X_i(c_i, q_i) \leq 0$ for any **IC** allocation rule. Thus, we have the $\widehat{\Pi}(\mathbf{x}, \boldsymbol{\rho}) \geq \Pi(\mathbf{x}, \bar{\boldsymbol{\rho}})$.

A proof technique identical to the one used to prove Theorem 3 establishes that $(\mathbf{x}^{\mathrm{I}}, \mathbf{t}^{\mathrm{I}})$ is an dominant strategy **IC** and **IR** procurement mechanism that maximizes the ironed-out buyer profit $\widehat{\Pi}$. Note that the corresponding offered surplus $\bar{\rho}^{\mathrm{I}}(q) \equiv 0$.

Suppose $\hat{K}_i(L(c_i, q_i), q_i) < K_i(L(c_i, q_i), q_i)$. Then Lemma 3 (d) implies that $H_i(c_i, q_i)$ is a constant for some neighborhood around c_i, i.e. $\partial_{c_i} X_i(c_i, q_i) = 0$ in some neighborhood of c_i. Consequently, the inequality (24) is an equality when $\mathbf{x} = \mathbf{x}^I$, i.e. $\widehat{\Pi}(\mathbf{x}^{\mathrm{I}}, \boldsymbol{\rho}^{\mathrm{I}}) = \Pi(\mathbf{x}^{\mathrm{I}}, \boldsymbol{\rho}^{\mathrm{I}})$. This establishes the result. ■

Theorem 4 characterizes the revenue maximizing direct mechanism when the virtual costs $H_i(c_i, q_i)$ satisfy semi-regularity, or equivalently, when the ironed-out virtual costs $\widehat{H}_i(c_i, q_i)$ satisfy regularity. When semi-regularity does not hold, the optimal direct mechanism can still be computed by numerically solving the stochastic program (21). Our numerical experiments lend support to the following conjecture.

Conjecture 5. *A revenue maximizing procurement mechanism has the following properties.*

1. *The side payments* $\bar{\boldsymbol{\rho}} \equiv \mathbf{0}$.
2. *There exist completely ironed-out virtual cost functions* $\tilde{H}_i$ *such that the corresponding point-wise solution* $\widetilde{\mathbf{x}} = \text{argmax}\left\{R(\sum_i \tilde{x}_i(\mathbf{b})) - \sum_{i=1}^n \tilde{H}_i(c_i, q_i)\tilde{x}_i(\mathbf{b}) : \mathbf{0} \leq \tilde{\mathbf{x}} \leq \mathbf{q}\right\}$ *is the revenue maximizing* **IC** *allocation rule.*
3. *The ironing procedure and the completely ironed-out virtual costs* $\tilde{H}_i(c_i, q_i)$ *depend on the* revenue function R, *in addition to the joint prior.*

Rochet and Chone (1998) present a general approach for multidimensional screening but in a model where the agents have both sided incentives i.e. they are allowed to mis-report their type both below and above it's true value.

3.3 Low-bid Implementation of the Optimal Auction

In this section, we assume that all the suppliers are identical, i.e. $F_i(c, q) = F(c, q)$, and that the distribution $F(c, q)$ satisfies Assumption 1 and Assumption 2. From (16) it follows that the expected transfer payment

$$T_i^*(c_i, q_i) = c_i X_i^*(c_i, q_i) + \int_{c_i}^{\bar{c}} X_i^*(u, q_i) du$$

Note that $T_i^*(c_i, q_i) = 0$, whenever $X_i^*(c_i, q_i) = 0$. Define a new point-wise transfer payment $\widetilde{\mathbf{t}}$ as follows.

$$\tilde{t}_i(\mathbf{b}) = \left(c_i + \frac{\int_{c_i}^{\bar{c}} X_i^*(t, q_i) dt}{X_i^*(c_i, q_i)} \right) x_i^*(c_i, q_i) \tag{25}$$

Then $\mathbb{E}_{(c_{-i}, q_{-i})}\left[t_i(c, q)\right] = T_i^*(c_i, q_i)$, therefore, $(\mathbf{x}^*, \tilde{\mathbf{t}})$ is Bayesian **IC** and **IR**. We use the transfer function $\widetilde{\mathbf{t}}$ to compute the bidding strategies in a "low bid" implementation of the direct mechanism. The "get-your-bid" auction proceeds as follows:

1. Supplier i bids the capacity $\hat{q}_i \leq q_i$, she is willing to provide and the marginal payment p_i she is willing to accept.
2. The buyer's action are as follows:

(a) Solve for the true marginal cost c_i by setting[1]

$$p_i = \phi(c_i, \hat{q}_i) = c_i + \frac{\int_{c_i}^{\bar{c}} X_i^*(t, \hat{q}_i)dt}{X_i^*(z, \hat{q}_i)}.$$

(b) Aggregates these bids and forms the virtual procurement cost function $\tilde{c}$ by setting

$$\tilde{c}(q) = \sum_{j=1}^{i-1} \hat{q}_{[j]} h_{[j]} + \Big(q - \sum_{j=1}^{i-1} \hat{q}_{[j]}\Big) h_{[i]} \tag{26}$$

for $\sum_{j=1}^{i-1} \hat{q}_{[j]} \leq q \leq \sum_{j=1}^{i} \hat{q}_{[j]}$, where as before $h_{[i]}$ denotes the i-th lowest virtual marginal cost and $\hat{q}_{[i]}$ is the capacity bid of the corresponding supplier.

(c) Solve for the quantity $\tilde{q} = \text{argmax}\{R(q) - \tilde{c}(q)\}$. Set the allocation

$$\tilde{x}_{[i]} = \begin{cases} \hat{q}_{[i]}, & \sum_{j=1}^{i} \hat{q}_{[j]} \leq \tilde{q}, \\ \tilde{q} - \sum_{j=1}^{i-1} \hat{q}_{[j]}, & \sum_{j=1}^{i-1} \hat{q}_{[j]} \leq q \leq \sum_{j=1}^{i} \hat{q}_{[j]}, \\ 0 & \text{otherwise.} \end{cases}$$

3. Supplier i produces $\tilde{x}_i$ and receives $\tilde{p}_i \tilde{x}_i$.

When all the suppliers are identical, the expected allocation function $X_i^*(c, q)$ is independent of the supplier index i. We will, therefore, drop the index.

Theorem 6. *The bidding strategy*

$$\begin{aligned} \tilde{q}(c, q) &= q, \\ \tilde{p}(c, q) &= \phi(c, q) \equiv c + \frac{\int_{c}^{\bar{c}} X^*(t, q)dt}{X^*(c, q)}, \end{aligned}$$

is a symmetric Bayesian Nash equilibrium for the "get-your-bid" procurement mechanism.

Proof: Comparing (14) and (26), it is clear that, in equilibrium, $\mathbf{x}^*(\mathbf{b}) = \widetilde{\mathbf{x}}(\mathbf{p}, \mathbf{q})$. Assume that all suppliers except supplier i use the bidding the proposed bidding strategy. Then the expected profit $\pi_i(\hat{p}_i, \hat{q}_i)$ of supplier i is given by

$$\begin{aligned} \pi_i(\hat{p}_i, \hat{q}_i) &= (\hat{p}_i - c_i)\tilde{X}_i(\hat{p}_i, \hat{q}_i) \\ &= (\hat{p}_i - c_i)X_i^*(\hat{c}_i, \hat{q}_i), \end{aligned}$$

[1] We assume that $\phi(c_i, q_i)$ is strictly increasing in c_i, for all q_i. This would be true, for example when the virtual costs H_i are strictly increasing in c_i. Note that previously, we had been working with allocations that were non-decreasing.

where $\hat{c}_i$ given by the solution of the equation $\hat{p}_i = \phi(c, \hat{q}_i)$. Thus, we have that

$$\begin{aligned}
\pi_i(\hat{p}_i, \hat{q}_i) &\leq (\hat{p}_i - c_i) X_i^*(\hat{c}_i, q_i), & (27)\\
&= \int_{\hat{c}_i}^{\bar{c}} X^*(u, q_i) du - (\hat{c}_i - c_i) X^*(c_i, q_i), \\
&= \int_{c_i}^{\bar{c}} X^*(u, q_i) du + \int_{\hat{c}_i}^{\bar{c}} X^*(u, q_i) du (\hat{c}_i - c_i) X^*(c_i, q_i), \\
&\leq \int_{c_i}^{\bar{c}} X^*(u, q_i) du = \pi(\tilde{p}(c_i, q_i), \tilde{q}(q_i)), & (28)
\end{aligned}$$

where (27) and (28) follows from, respectively, Lemma 2 (b) and (a). Thus, it is optimal for supplier i to bid according to the proposed strategy. ■

3.4 Corollaries

Since the point-wise profit in (12) depends on the capacity q_i only through the conditional distribution $F_i(c_i|q_i)$ of the cost c_i given capacity q_i, the following result is immediate.

Corollary 1. *Suppose the marginal cost c_i and capacity q_i are independently distributed. Then the optimal allocation rule (and the corresponding transfer function) is insensitive to the capacity distribution.*

Contrasting this result with the "get-your-bid" implementation in the last section, we find that although the optimal auction mechanism is insensitive to the capacity, the supplier bidding strategies may depend on the capacity distribution.

The following result characterizes the buyers profit function when the suppliers' capacity is common knowledge.

Corollary 2. *Suppose suppliers' capacity is common knowledge. Then the buyers expected profit under any feasible,* **IC** *and* **IR** *allocation rule* $\mathbf{x}$ *is given by*

$$\Pi(\mathbf{x}) = \mathbb{E}_c \left[R\left(\sum_{i=1}^{n} x_i(c) \right) - \sum_{i=1}^{n} x_i(c) \left(c_i + \frac{F_i(c_i)}{f_i(c_i)} \right) \right] \tag{29}$$

Suppose the buyer wishes to procure a fixed quantity Q from the suppliers. Since a given realization of the capacity vector $\mathbf{q}$ can be insufficient for the needs to the buyer, i.e. $\sum_{i=1}^{n} q_i < Q$, we have to allow for the possibility of an exogenous procurement source. We assume that the buyer is able to procure an unlimited quantity at a marginal cost $c_0 > \bar{c}$. Let $EC(Q)$ denote the expected cost of procuring quantity Q by any optimal mechanism.

Corollary 3 (**Fixed Quantity Auction**). *Suppose Assumption 1 and 2 hold. Then*

$$EC(Q) = \mathbb{E}_{(c,q)} \left\{ \begin{array}{rl} \min & \sum_{i=1}^n x_i(c,q) q_i H_i(c_i, q_i) + q_0 c_0 \\ s.t. & \sum_{i=1}^n x_i q_i + q_0 = Q \\ & 0 \le \mathbf{x} \le q \end{array} \right\}. \tag{30}$$

Results in this paper can be adapted to other principle-agent mechanism design settings. Consider monopoly pricing with capacitated consumers. Suppose the monopolist seller with a strictly convex production cost $c(x)$ faces a continuum of customers with utility of the form

$$u(x,t;\theta,q) = \begin{cases} \theta x - t, & x \le q, \\ -\infty, & x > q, \end{cases}$$

where (θ, q) is the private information of the consumers. The form of the utility function $u(x,t;\theta,q)$ prevents the customer from overbidding its capacity to consume. This is necessary for the seller to be able to check individual rationality. As always the type distribution $F : [\underline{\theta}, \bar{\theta}] \times [\underline{q}, \bar{q}] \to \mathbb{R}_{++}$ is common knowledge.

Corollary 4. *Suppose the distribution $F(\theta,q)$ satisfies the regularity assumption that $\nu(\theta,q) = \theta - \frac{1-F(\theta|q)}{f(\theta|q)}$ is separately non-decreasing in both θ and q. Then the following holds for monopoly pricing with capacitated buyers.*

1. *The seller profit $\Pi(x)$ for any feasible,* **IC** *allocation rule x, the seller expected profit is of the form*

$$\Pi(x) = \mathbb{E}_{(\theta,q)} \left[\Big(\theta - \frac{1-F(\theta|q)}{f(\theta|q)} \Big) x(\theta,q) - c(x(\theta,q)) \right]$$

2. *An optimal direct mechanism is given by the allocation rule*

$$x^*(\theta,q) = \underset{0 \le x \le q}{argmax} \left[\Big(\theta - \frac{1-F(\theta|q)}{f(\theta|q)} \Big) x - c(x) \right]$$

and transfer payment

$$t^*(\theta,q) = \int_{\underline{\theta}}^{\theta} x^*(t,q) dt$$

Since the type space is two-dimensional, the optimal direct mechanism can be implement by a posted tariff only if the parameter θ and the capacity q are independently distributed.

All our results in this section easily extend to nonlinear convex production cost $c_i(\theta, x)$, $\theta \in [\underline{\theta}, \bar{\theta}]$, that are super-linear, i.e. $\frac{\partial^2 c_i}{\partial\theta\partial x} > 0$. In this case, the virtual production cost $H_i(\theta_i, x)$ is given by

$$H_i(\theta_i, x) = c_i(\theta_i, q_i, x) + c_{i\theta}(\theta_i, x)\frac{F_i(\theta_i|q_i)}{f_i(\theta_i|q_i)}.$$

4 Multiple Component Model

We next consider a model for procuring variable quantities of multiple products (or components). Consider a situation where a buyer (typically a large manufacturing corporation or large retail chain like Walmart) wants to purchase m different product types. The expected revenue of the buyer as a function of quantity procured is denoted by $R : \mathbb{R}^m \to \mathbb{R}$. The production cost incurred by supplier i in producing the bundle $\mathbf{x} \in \mathbb{R}^m_+$ is given by $c^i(\theta_i, \mathbf{x})$, where $\theta_i \in [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$ is privately known. The restriction that the production cost is parameterized by a single parameter θ is required to keep the problem tractable. However, we allow general cost functions in order to model asymmetry in production costs.

Assumption 4. *The revenue function R and the production cost function $c^i(\theta_i, \mathbf{x})$, $i = 1, \ldots, n$, satisfy the following regularity conditions.*

i) The revenue function R is twice continuously differentiable, concave and supermodular[2]. In addition, for all $j = 1, \ldots, m$, $\frac{\partial R(\mathbf{x})}{\partial x_j} \to 0$ as $x_j \to \infty$ and $\frac{\partial R(\mathbf{x})}{\partial x_j} \to \infty$ as $x_j \to 0$.

ii) The production cost function $c^i(\theta, \mathbf{x})$ satisfies $c^i_\theta(\theta, \mathbf{x})$[3] > 0, $\nabla_x c^i(\theta, \mathbf{x}) \geq \mathbf{0}$, $\nabla_x c^i_\theta(\theta, \mathbf{x}) \geq \mathbf{0}$, $c^i_{\theta\theta}(\theta, \mathbf{x}) \geq 0$, $\mathbf{c}^i_{xx}(\theta, \mathbf{x}) \succeq 0$ and $\mathbf{C}^i_{\theta xx}(\theta, \mathbf{x}) \succeq 0$, where $\mathbf{M} \succeq \mathbf{0}$ denotes that the matrix $\mathbf{M}$ is symmetric positive semidefinite.

Since the revenue function $R(\mathbf{x})$ is supermodular, we are implicitly assuming that the bundle $\mathbf{x}$ has complementary products.

We assume that each supplier has fixed capacity for each product. Let $\mathbf{a}_i$ denote the capacity vector of supplier i. Then an allocation $\mathbf{x}_i$ to supplier i is feasible only if $\mathbf{x}_i \leq \mathbf{a}_i$. The capacity $\mathbf{a}_i$ exogenously given so that supplier do not have an option of acquiring additional capacity. The capacity vectors $\mathbf{a}_i \in \mathbf{A}$ where $\mathbf{A}$ is a convex subset of $\mathbb{R}^m_+$. The prior distribution $f_i : [\underline{\theta}, \bar{\theta}] \times \mathbf{A} \to \mathbb{R}_{++}$ of the supplier i type $(\theta_i, \mathbf{a}_i)$ is common knowledge. The following regularity assumption on the prior distribution is analogous to Assumption 1 in the previous section.

Assumption 5. *The prior density f_i has full support. Thus, the conditional density $f_i(\theta|\mathbf{a})$ has full support for all $\mathbf{a} \in \mathbf{A} \subset \mathbb{R}^m_+$.*

[2] Recall that a twice continuously differentiable function $f : \mathbb{R}^m \to \mathbb{R}$ is supermodular if the cross derivatives $\frac{\partial 2R}{x_i x_j} \geq 0$ $i \neq j$.

[3] $c^i_\theta(\theta, x)$ denote the partial derivative of the production costs of supplier i with respect to θ at (θ, x) and so on.

We take the direct mechanism approach to construct the optimal direct mechanism. Thus, supplier i's bid $\hat{\mathbf{b}}_i$ is of the form $\hat{\mathbf{b}}_i = (\hat{\theta}_i, \hat{\mathbf{a}}_i) \in [\underline{\theta}, \bar{\theta}] \times \mathbf{A}$. As in the case in the previous section, we assume overbidding capacity results in a heavy penalty. Therefore, the supplier i capacity bid $\hat{\mathbf{a}}_i$ must satisfy $\hat{\mathbf{a}}_i \leq \mathbf{a}_i$, where the inequality is interpreted component-wise. Let $\hat{\mathbf{b}} = [\hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_n]$ and let $\mathbf{B} = \left([\underline{\theta}, \bar{\theta}] \times \mathbf{A}\right)^n$ denote the type-space of all the suppliers.

1. an allocation rule $\mathbf{x} : \mathbf{B} \mapsto \mathbb{R}^{m \times n}$ that for each bid $\hat{\mathbf{b}} \in \mathbf{B}$ specifies the bundle $\mathbf{x}_i \in \mathbb{R}^m$ to be ordered from each of the suppliers, and

2. a transfer payment function $\mathbf{t} : \mathbf{B} \rightarrow \mathbb{R}^n$ that maps each bid vector $\hat{\mathbf{b}}$ to the monetary transfer from the buyer to the suppliers.

The buyer seek an allocation function $\mathbf{x}$ and a transfer function $\mathbf{t}$ that maximizes the ex-ante expected profit

$$\Pi(\mathbf{x}, \mathbf{t}) \equiv \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^{n} \mathbf{x}_i(\mathbf{b})\right) - \sum_{i=1}^{n} t_i(\mathbf{b})\right]$$

subject to **IC**, and **IR**, and feasibility, i.e. $\mathbf{x}_i(\mathbf{b}) \leq \mathbf{a}_i$. The following result is the analog of Lemma 1 for multi-component markets.

Lemma 4. *All multi-component procurement auctions satisfy the following.*

1. *A feasible allocation rule $\mathbf{x} : \mathbf{B} \rightarrow \mathbb{R}^n_+$ is* **IC** *if, and only if, for all $i = 1, \ldots, n$,*

$$C^i(\theta_i, \mathbf{a}_i) \equiv \mathbb{E}_{\mathbf{b}_{-i}} c^i_\theta(\theta_i, \mathbf{x}(\mathbf{b}))$$

is non-increasing in θ_i.

2. *A mechanism $(\mathbf{x}, \mathbf{t})$ is* **IC** *and* **IR** *if, and only if, the allocation rule $\mathbf{x}$ satisfies (a) and the offered surplus $\rho_i(\theta_i, \mathbf{a}_i)$ is of the form*

$$\rho_i(\theta_i, \mathbf{a}_i) = \overline{\rho}_i(\mathbf{a}_i) + \int_{\theta_i}^{\bar{\theta}} C^i_\theta(z, \mathbf{a}_i) dz \tag{31}$$

with $\rho_i(\theta_i, \mathbf{a}_i)$ non-negative and non-decreasing in a_{ij} for all $\mathbf{a}_{-j}$ and θ.

The proof of this result is very similar to that of Lemma 1 and is, therefore, left to the reader to reconstruct.

The fact that the cost parameter θ is one dimensional is crucial for Lemma 4 to hold. All allocation rule which are incentive compatible with respect to the cost parameter θ_i can be made incentive compatible with respect to the capacity simply by constructing appropriate side payments $\bar{\rho}_i$ which are independent of θ_i. The following analog of Theorem 1 can be easily established using Lemma 4.

Theorem 7. *Suppose Assumption 5 holds. Then the buyers profit corresponding to any allocation rule* $\mathbf{x}$ *that satisfies* **IC** *and* **IR** *is given by*

$$\Pi(\mathbf{x}, \bar{\boldsymbol{\rho}}) = \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^{n} \mathbf{x}_i(\mathbf{b})\right) - \sum_{i=1}^{n} H^i(\theta_i, \mathbf{a}_i, \mathbf{x}_i) - \sum_{i=1}^{n} \bar{\rho}_i(\mathbf{a}_i)\right]$$

where $\bar{\rho}_i$ *are* θ_i*-independent side payments, and*

$$H^i(\theta_i, \mathbf{a}_i, \mathbf{x}_i) \equiv c^i(\theta, \mathbf{x}) + c^i_\theta(\theta, \mathbf{x})\frac{F_i(\theta|\mathbf{a})}{f_i(\theta|\mathbf{a})} \tag{32}$$

are virtual production costs.

Thus, the mechanism design problem reduces to the following optimization problem.

$$\begin{array}{lll} \max_{\mathbf{x}(\theta,\mathbf{a}),\bar{\rho}_i(\mathbf{a}_i)} & \mathbb{E}_{\mathbf{b}}\left[R\left(\sum_{i=1}^n \mathbf{x}_i(\mathbf{b})\right) - \sum_{i=1}^n H^i(\theta_i, \mathbf{a}_i, \mathbf{x}_i) - \sum_{i=1}^n \bar{\rho}_i(\mathbf{a}_i)\right], & \\ \text{s.t.} & \mathbf{x}_i \leq \mathbf{a}_i, & \text{for all } i = 1, \ldots, n, \\ & C^i_\theta(\theta_1, \mathbf{a}) \leq C^i_\theta(\theta_2, \mathbf{a}) & \text{for all } \theta_1 \geq \theta_2, \theta_1, \theta_2 \in [\underline{\theta}, \bar{\theta}], \mathbf{a} \in \mathbf{A}, i = 1, \ldots, n \\ & \bar{\rho}_i(\mathbf{a}_1) \geq \bar{\rho}_i(\mathbf{a}_2) & \mathbf{a}_1 \geq \mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}, i = 1, \ldots, n. \end{array}$$

Next, we discuss conditions under which the point-wise allocation rule

$$\mathbf{x}^*(\mathbf{b}) \in \underset{0 \leq \mathbf{x}_i \leq \mathbf{a}_i}{\operatorname{argmax}} \left\{R\left(\sum_{i=1}^{n} \mathbf{x}_i(\theta, a)\right) - \sum_{i=1}^{n} H_i(\theta_i, \mathbf{a}_i, \mathbf{x}_i)\right\} \tag{33}$$

is the revenue maximizing procurement mechanism. Recall that in the context of the single component model, in order for the point-wise optimal allocation rule to be the revenue maximizing allocation rule, one needed to ensure that $X_i^*(c_i, q_i)$ is non-increasing in c_i and non-decreasing in q_i. We need a similar condition here.

Assumption 6. *The virtual cost function* $H^i(\theta_i, \mathbf{a}_i, \mathbf{x}_i)$ *defined in (32) satisfies the following regularity conditions.*

1. *For all* $\mathbf{b}_i = (\theta_i, \mathbf{a}_i)$*, the virtual cost function* $H^i(\theta_i, \mathbf{a}_i, \mathbf{x})$ *is sub-modular in* $\mathbf{x}$*.*
2. *The gradient* $\nabla_x H^i(\theta, \mathbf{a}, \mathbf{x})$ *satisfies the following:*
   - *(a)* $\nabla_x H^i(\theta_2, \mathbf{a}, \mathbf{x}) \geq \nabla_x H^i(\theta_1, \mathbf{a}, \mathbf{x})$ *for all* $\theta_2 \geq \theta_1$*.*
   - *(b)* $\nabla_x H^i(\theta, (a_j, \mathbf{a}_{-j}), \mathbf{x}) \leq \nabla_x H^i(\theta, (a'_j, \mathbf{a}_{-j}), \mathbf{x})$ *for all* $a_j \geq a'_j$*.*

Note that we are implicitly assuming that the virtual cost function $H^i(\mathbf{b}, \mathbf{x})$ is differentiable in $\mathbf{x}$. Assumption 6 (i) is satisfied when production cost are separable, i.e. $\frac{\partial^2 c_i(\theta,x)}{\partial x_i \partial x_j} = 0$. If the capacity vector $\mathbf{a}_i$ and the cost parameter θ_i are independently distributed, then Assumption 6 (ii) is satisfied when conditional density satisfy $\frac{F_i(\theta|a)}{f_i(\theta|a)} = \frac{F_i(\theta)}{f_i(\theta)}$ is monotone in θ and $\nabla_x c^i_{\theta\theta} \geq 0$.

Theorem 8. *Suppose Assumptions 4, 5 and 6 hold and revenue function is linear*[4]*. Define transfer payments*

$$t_i^*(\mathbf{b}) = c^i(\theta_i, \mathbf{x}_i^*) + \int_{\theta_i}^{\bar{\theta}} c_\theta^i(u, \mathbf{x}_i^*)du, \qquad i = 1, \ldots, n. \tag{34}$$

Then $(\mathbf{x}^*, \mathbf{t}^*)$ *is a revenue maximizing multi-component procurement auction under which bidding truthfully forms a dominant strategy equilibrium.*

Proof: From the proof technique used to establish the single-component result in Theorem 3, it follows that all we need to show is that

1. $c_\theta^i\big(\theta_i, \mathbf{x}^*((\theta_i, \mathbf{a}_i), \mathbf{b}_{-i})\big)$ is non-increasing in θ_i for all $\mathbf{a}_i$ and $\mathbf{b}_{-i}$, and
2. $c_\theta^i\big(\theta_i, \mathbf{x}^*((\theta_i, \mathbf{a}_i), \mathbf{b}_{-i})\big)$ is non-decreasing in a_{ij} for all θ_i, a_{ik}, $k \neq j$, and $\mathbf{b}_{-i}$.

We use the following result.

Theorem 9 (Milgrom and Shannon (1994), Theorem 4′)**.** *Let* $f : \mathbf{X} \times \mathbf{T} \to \mathbb{R}$*, where* $\mathbf{X}$ *is a lattice and* $\mathbf{T}$ *is a partially ordered set. If* $\mathbf{S} : \mathbf{T} \to \mathbf{2}^{\mathbf{X}}$ *is nondecreasing and if* f *is quasi-supermodular in* $\mathbf{x}$ *and satisfies the single crossing property in* $(\mathbf{x}, \mathbf{t})$*, the every selection* $\mathbf{x}^*(\mathbf{t})$ *from argmax*$_{\mathbf{x}\in\mathbf{S}(\mathbf{t})} f(\mathbf{x}, \mathbf{t})$ *is monotone nondecreasing in* $\mathbf{t}$*.*

Define

$$\begin{aligned}
\mathbf{T} &= \Big([-\bar{\theta}, -\underline{\theta}] \times \mathbf{A}\Big)^n, \\
\mathbf{X} &= \Big(\mathbb{R}^m\Big)^n \\
\mathbf{S}(t) &= \{\mathbf{x} \mid \mathbf{0} \le \mathbf{x}_i \le \mathbf{a}_i, \forall i = 1, \ldots, n\}, \\
f(\mathbf{x}, \mathbf{b}) &= R\Big(\sum_{i=1}^n \mathbf{x}_i\Big) - \sum_{i=1}^n H^i(\mathbf{b}_i, \mathbf{x}_i),
\end{aligned}$$

The ordering for both the lattice $\mathbf{X} \subset \prod_{i=1}^n \mathbb{R}^{m+1}$ and the partially-ordered set $\mathbf{T} \subseteq \prod_{i=1}^n \mathbb{R}^n$ is the usual product ordering.

It is clear that $\mathbf{S}(t)$ is "box-shaped" and is increasing in each of the component of a_{ij} in the usual strong order on $\mathbf{R}^m$. By Assumption 4 (i), Assumption 6 (i) and the linearity of R, it follows that $f(\mathbf{x}, \mathbf{b})$ is quasi-supermodular in $\mathbf{x}$ for all $\mathbf{b} \in \mathbf{T}$.

[4]This theorem holds either $n = 1$ or $R(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + b$ for some $\mathbf{a} \gg 0, b \ge 0$ or $n = 2$ and $R(\mathbf{x}) = \sum_{j=1}^m R_j(x_j)$.

Let $\mathbf{x}^2 \geq \mathbf{x}^1$ and $\mathbf{b}^2 \geq \mathbf{b}^1$. Then

$$
\begin{aligned}
f(\mathbf{x}^2, \mathbf{b}^2) - f(\mathbf{x}^1, \mathbf{b}^2) &= \Big(R\Big(\sum_{i=1}^{n} \mathbf{x}_i^2\Big) - R\Big(\sum_{i=1}^{n} \mathbf{x}_i^1\Big)\Big) - \sum_{i=1}^{n} \left(H^i(\mathbf{b}_i^2, \mathbf{x}_i^2) - H^i(\mathbf{b}_i^2, \mathbf{x}_i^1)\right), \\
&= \Big(R\Big(\sum_{i=1}^{n} \mathbf{x}_i^2\Big) - R\Big(\sum_{i=1}^{n} \mathbf{x}_i^1\Big)\Big) \\
&\quad - \sum_{i=1}^{n} \int_0^1 \nabla_x H^i(\mathbf{x}_i^1 + (\mathbf{x}_i^2 - \mathbf{x}_i^1)u, \mathbf{b}^2)^T (\mathbf{x}_i^2 - \mathbf{x}_i^1) du, \\
&\geq \Big(R\Big(\sum_{i=1}^{n} \mathbf{x}_i^2\Big) - R\Big(\sum_{i=1}^{n} \mathbf{x}_i^1\Big)\Big) \\
&\quad - \sum_{i=1}^{n} \int_0^1 \nabla_x H^i(\mathbf{x}_i^1 + (\mathbf{x}_i^2 - \mathbf{x}_i^1)u, \mathbf{b}^1)^T (\mathbf{x}_i^2 - \mathbf{x}_i^1) du, \qquad (35) \\
&= f(\mathbf{x}^2, \mathbf{b}^1) - f(\mathbf{x}^1, \mathbf{b}^1),
\end{aligned}
$$

where (35) follows from Assumption 6 (ii). Thus, it follows that $f(\mathbf{x}, \mathbf{b})$ satisfies the single-crossing property in $(\mathbf{x}, \mathbf{b})$.

Consequently, Theorem 9 implies that $\mathbf{x}^*(\mathbf{b})$ is monotonically non-decreasing in $\mathbf{b}$. Since the order of the θ-component of the type vector was reversed (see the definition of $\mathbf{T}$), we have that $\mathbf{x}^*(\mathbf{b})$ is non-increasing in θ_i and non-decreasing in the capacity $\mathbf{a}_i$. The result now follows from the regularity conditions in Assumption 4-(ii). ■

4.1 Auctioning Multi-period Supply Chain Contract

We apply the multi-product model to the setting where a manufacturer wants to procure units of a single good over multiple periods. We assume full commitment and do not allow for renegotiation. This assumption is reasonable in high-value industries where the end-consumers, even though elastic in their consumption at the time of contract negotiation, would expect agreed-upon deliveries at the time of actual consumption.

Suppose the time horizon of the contracts consists of w periods. The production vector $\mathbf{x}_i$ of supplier i is of the form $\mathbf{x}_i = (x_{i1}, \ldots, x_{iw}) \in \mathbb{R}_+^w$, where x_{it} denotes the production in period t. The full-commitment assumption implies that the vector $\mathbf{x}_i$ can be treated as a bundle of T different products with production cost $c^i(\theta_i, \mathbf{x}_i)$. The capacity constraints for the supplier i are modeled as before: $\mathbf{x}_i \leq \mathbf{a}_i$. This capacity model implicitly assumes that the suppliers' production is limited due to some exogenous effects other than resources.

Suppose the cost function $c^i(\theta_i, \mathbf{x}_i)$, the revenue function $R : \mathbb{R}_+^w \to \mathbb{R}_+$ and the prior distributions $f_i(\theta_i, \mathbf{a}_i)$ satisfy conditions of Theorem 8, i.e. we are in the so-called "regular" case. Then the optimal allocation $\mathbf{x}$ is given by the solution of the point-wise optimization problem (33) and this allocation is be implemented by the transfer payments $\mathbf{t}$ defined in (34).

Suppose the cost function $c^i(\theta_i, \mathbf{x})$ and revenue function $R(\mathbf{x})$ are both separable in $\mathbf{x}$, i.e. the quantities procured and sold in different time periods do not interact. Then point-wise optimization problem is separable. Thus, the reverse auction reduces to m single-component auctions – one for each time period. These auctions can be either be held sequentially at the beginning of each period or all together at time zero.

5 Conclusion and Extensions

We presented a procurement mechanism that is able to optimally screen for both privately known capacities and privately known cost information. The results can be easily adapted to other principle-agent mechanism design problems in which agents have a privately known bounds on consumption.

We explicitly characterize the optimal procurement mechanism when the costs and the prior distribution satisfy the so-called "regularity" condition - a form of negative affiliation between capacities and costs (see Assumption 2). Under regularity, the private information about capacities works to the advantage of the buyer as it reveals some information about the costs. We show that the optimal mechanism cannot be implemented as uniform price auction, e.g. K^{th} price auction. We also show that the two-dimensional private information forbids a $P-Q$ curve indirect implementation (see, e.g. Deshpande and Schwarz (2005); Chen (2004)) of the optimal direct mechanism.

In the absence of regularity, there are complex tradeoffs between incentives to reveal the capacity and the incentives to reveal cost. Consequently, the optimal auction mechanism is a solution to complex stochastic program. In this case, we are only able to characterize the mechanism when a certain "semi-regularity" condition holds.

Our model can be easily extended to multi-product case with complementarities when the private cost information is single dimensional. An application of this model in auctioning multi-period supply contract also showed that a buyer with full commitment who faces the inter-temporal risk of variable capacities over time from the cost efficient supplier can effectively hedge this risk by committing in the beginning to order from different suppliers in different periods.

Some natural extension of this works as follows:

1. Suppliers can ex-post purchase additional capacity q^a at a cost $g(q^a)$. In this case, the optimal supplier surplus can be written as

$$\pi_i(c_i, q_i) = \underset{(\hat{c}_i, \hat{q}_i)}{\text{argmax}} \{T_i(\hat{c}_i, \hat{q}_i) - c_i X_i(\hat{c}_i, \hat{q}) - C_i^q(c_i, q_i)\} \tag{36}$$

where

$$C_i^q(c_i, q_i) = \mathbb{E}_{(c_{-i},q_{-i})}\left[\mathbf{1}\left(x_i((\hat{c}_i, c_{-i}), (\hat{q}_i, q_{-i})) > q_i\right) g_i\big(x_i((\hat{c}_i, c_{-i}), (\hat{q}_i, q_{-i})) - q_i\big)\right]$$

Unlike in (1), the true capacity q explicitly appears in (36). This makes the problem truly 2-dimensional and Lemma 1 fails to hold.

2. Multi-product model with multidimensional private information about the product cost: Even the uncapacitated version of this problem remains unsolved.

3. Construct more robust implementation of the optimal direct mechanism: the possibility of using dynamic games of incomplete information which converge the equilibrium of the one shot game.

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A Uncapacitated suppliers with two-dimensional type

The production cost of a capacitated supplier is of the form

$$c_i(\theta_i, q_i, x) = \begin{cases} \theta_i x, & 0 \le x \le q_i, \\ +\infty, & x > q_i, \end{cases}$$

where the function $c(\cdot)$ is common knowledge but the parameters (θ_i, q_i) are privately known. Thus, the supplier has a two-dimensional type. Alternatively, we can model the cost function c of capacitated suppliers as follows.

$$c(\theta_i, \gamma_i, x) = \theta_i x + \gamma_i x^2,$$

where $\theta_i \in [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$ and $\gamma_i \in [\underline{\gamma}, \bar{\gamma}] \subset \mathbb{R}_+$. In this model, let $\pi_i(\hat{\theta}_i, \hat{\gamma}_i; \theta_i, \gamma_i)$ denotes the surplus when supplier i with type (θ_i, γ_i) bids $(\hat{\theta}_i, \hat{\gamma}_i)$. Since $c(\theta_i, \gamma_i, x)$ is a smooth function, the **IC** conditions simplify to

$$\left.\frac{\partial \pi_i(\hat{\theta}_i, \gamma_i)}{\partial \hat{\theta}_i}\right|_{\hat{\theta}_i = \theta_i} = 0 \qquad \left.\frac{\partial \pi_i(\theta_i, \hat{\gamma}_i)}{\partial \hat{\gamma}_i}\right|_{\hat{\gamma}_i = \gamma_i} = 0 \qquad \forall \theta_i \in (\underline{\theta}, \bar{\theta}), \gamma \in (\underline{\gamma}, \bar{\gamma})$$

These **IC** conditions lead to the following envelope conditions

$$\frac{\partial \pi_i(\theta_i, \gamma_i)}{\partial \theta_i} = -X_i(\theta_i, \gamma_i) \qquad \frac{\partial \pi_i(\theta_i, \gamma_i)}{\partial \gamma_i} = -\mathbb{E}_{(\theta_{-i}, \gamma_{-i})} x_i^2(\theta_i, \gamma_i) \tag{37}$$

An allocation rule $\mathbf{x}$ is **IC**, i.e. there exists a surplus function π satisfying (37) only if

$$\oint_C \begin{pmatrix} -X_i(\theta_i, \gamma_i) \\ -\mathbb{E}_{(\theta_{-i}, \gamma_{-i})} x_i^2(\theta_i, \gamma_i) \end{pmatrix}^T d\mathbf{l} = 0 \tag{38}$$

for all closed smooth paths $C \subset [\underline{\theta}, \bar{\theta}] \times [\underline{\gamma}, \bar{\gamma}]$ and all suppliers i. When (38) holds, the surplus $\pi_i(\theta_i, \gamma_i)$ can be written as

$$\pi_i(\theta_i, \gamma_i) = \pi_i(\bar{\theta}, \bar{\gamma}) + \int_l \nabla \pi_i dl \tag{39}$$

where $\nabla \pi_i = \begin{pmatrix} -X_i(\theta_i, \gamma_i) \\ -\mathbb{E}_{(\theta_{-i}, \gamma_{-i})} x_i^2(\theta_i, \gamma_i) \end{pmatrix}$ represent the gradient of the expected surplus function as defined in (37) and integration is over *any* path joining $(\bar{\theta_i}, \bar{\gamma_i})$ and (θ_i, γ_i). Integrating along $(\bar{\theta_i}, \bar{\gamma_i}) \to (\theta_i, \bar{\gamma_i}) \to (\theta_i, \gamma_i)$, we get

$$\pi_i(\theta_i, \gamma_i) = \pi(\bar{\theta}, \bar{\gamma}) + \int_{\theta}^{\bar{\theta}} X_i(t, \bar{\gamma}) dt + \int_{\gamma}^{\bar{\gamma}} \mathbb{E}_{(\theta_{-i}, \gamma_{-i})} x_i^2(\theta_i, t) dt \tag{40}$$

Interchanging the order of integrals (assuming some second-order regularity conditions), the buyer's expected profit from any **IC** allocation rule $\mathbf{x}$ can be written as

$$\begin{aligned} \Pi(\mathbf{x}) = & \mathbb{E}_{(\theta, \gamma)} \left[R\left(\sum_{i=1}^{n} x_i(\theta, \gamma) \right) - \sum_{i=1}^{n} \left(\theta_i + \frac{F_i(\theta|0)}{f(\theta|0)} \right) x_i(\theta, \gamma) \right. \\ & \left. - \sum_{i=1}^{n} \left(\gamma_i + \frac{F(\gamma_i|\theta_i)}{f(\gamma_i|\theta_i)} \right) x_i^2(\theta_i, \gamma_i) \right] \end{aligned}$$

Thus, the buyer's optimization problem reduces to selecting a feasible allocation rule $\mathbf{x}$ which maximize the expected profit function $\Pi(\mathbf{x})$ given above subject to the integrability conditions (38). This is an optimal control problem and computing its optimal solution is likely to be very hard (see Rochet and Stole (2003) and Rochet and Chone (1998)).