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Many-to-one Stable Matching: Geometry and Fairness

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Abstract

The stable marriage problem and its variants have been studied extensively over the last few decades. Beginning with the pioneering work of Gale and Shapley in the 60s, and through the work of Roth since the 80s, the elegant theory of this model has been used repeatedly in the design (and redesign) of two-sided labor markets (for instance, the National Resident Matching Program (NRMP) in the US, where medical interns are assigned to hospitals through a centralized matching mechanism).

Our focus in this paper is on the many-to-one stable matching problem, also called the college admissions problem. It has long been recognized that the (natural) stable matching mechanisms suggested by the proposal algorithm of Gale and Shapley compute the best stable matching for one side of the market, which is the worst stable matching for the the other side of the market. The resulting controversy has recently led to the re-design of the matching mechanism in the NRMP recently.

Motivated by the need to design "fair" stable matching mechanisms, we use a recent characterization of the stable admissions polytope (Baiou and Balinski (2000)) to provide a geometric structure to its fractional solutions. We use this structure to propose a fair stable matching mechanism for the many-to-one stable matching problem: assign each student to her *median* university, and each university to its *median set* of students. Several structural results for the stable admissions problem and its variants also follow naturally from this geometry.

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1 Introduction

The stable marriage problem and its variants have been studied extensively over the last few decades. Beginning with the pioneering work of Gale and Shapley [13], this class of problems has captured the attention of researchers and practitioners in several disciplines such as computer science, economics, mathematics, and operations research. The multidisciplinary nature of these problems has led to a thorough understanding of many aspects such as the design of efficient algorithms to find a stable matching, the structure of all solutions to a given stable matching instance, etc. In fact, by now there are several well-developed approaches to studying stable matching problems: the combinatorial/algorithmic approach as summarized in the books of Knuth [17] and Gusfield & Irving [14]; the linear programming approach initiated by Vande Vate [31], and further developed by Rothblum [28], Roth, Rothblum & Vande Vate [26], Teo & Sethuraman [30], Baiou & Balinski [5] and Fleiner [12]; the fixed-point approach of Subramanian [29] and Feder [9, 10]; an alternative fixed-point approach of Adachi [1]; yet another fixed-point approach due to Fleiner [11]; and a graph-theoretic approach due to Baiou and Balinski [4].

In addition to its elegant theory, a particularly appealing feature of the stable matching model is its applicability. In fact, starting with the work of Roth [20], applications to the National Resident Matching Program (NRMP) and related labor markets have given rise to interesting theoretical questions, resulting in a better understanding of the area. Moreover, insights from this theory have been useful in the redesign of the stable matching algorithm used by the NRMP [21]. An introduction to the theory of stable matchings with particular emphasis on applications to labor markets is elegantly summarized in Roth & Sotomayor [25]; recent work in this direction include Roth & Peranson [23, 24], Cantala [8], and Klaus & Kiljn [16].

This close interaction between applications and theory continues to this day, and is an inspiration for the questions we study, described next.

Problem description, Motivation, and Results. Our focus in this paper is mostly on the many-to-one stable matching problem, also called the stable admissions problem. Our interest in this model is because of its ability to capture most real-life labor markets, which are inherently many-to-one. Occasionally, many-to-many stable matching models arise in practice; for example, medical interns in the UK need to be assigned to two hospitals. Even though the many-to-many stable matching problem is not our main object of study, some of our results generalize to this setting as well, and we shall indicate this whenever appropriate. Our work is motivated by two concerns that arise in the implementation of stable matching mechanisms in practice: first, the design of a fair stable matching mechanism; and second, modifying the stable matching mechanism to deal with couples who prefer to be assigned to hospitals (or universities or firms) in the same geographical area.

It has long been recognized that the (natural) stable matching mechanisms suggested by the "proposal" algorithm of Gale & Shapley compute the *best* stable matching for *one* side of the market, which is the worst stable matching for the other side of the market. From its inception until 1997, the NRMP used the hospital-optimal matching mechanism as the basis of its allocation. However, increasing pressure from student bodies and others[1], prompted by the fact that the hospital-optimal solution is the worst for the students, led to the redesign of the matching mechanism in 1998 [23].

The inequitable treatment of the participants, on different side of the markets, persists in most known matching mechanisms. This has motivated the need for the design of fair stable matching mechanisms, which do not overtly favor one side of the market over the other. Considerable efforts have been devoted to finding fair stable matchings, including the *egalitarian* solution (minimize the sum of the ranks of the participants) and the *minimum-regret* solution (maximize the welfare of the participant worst-off in the matching) [14]. In contrast to this cardinal approach, we can also focus on the ordinal aspects of the model, and consider probabilistic stable matching mechanisms that find a probability distribution over the set of stable matchings. Whenever an agent has the same probability to move at a certain point in the procedure that determines the final probability distribution, the random stable matching mechanism is considered to be "procedurally fair". Klaus and Klijn [16] analyzed the random matching mechanisms introduced in Aldershof et al. [3], Roth and Vande Vate [27] and Ma [18] and constructed the associated probability distribution on the set of stable matching solutions with these random matching mechanisms. Unfortunately, these random matching mechanisms could not ensure that the probability distribution on the set of solutions is uniformly distributed, since that not all stable matching solutions can be constructed through the random matching mechanism with non-zero probability. In fact, Klaus and Klijn [16]

[1] This took its clearest form in an exchange in the June 1995 issue of Academic Medicine, followed by position papers put out by the American Medical Students Association and others. Much of that discussion was oriented around results concerning "two-sided matching markets," of which the NRMP is a special case. The dominant solution concepts in this area are the student-optimal matching solution and the university-optimal matching solution. However, neither of these solution concepts was deemed fair and acceptable to both the students and the hospitals.

constructed an example and showed that a stable matching that constitutes a perfect compromise between contrary preferences on both sides of the market may never result from this class of random matching mechanisms. Curiously, the perfect compromise solution identified there is the *median* solution concept (where each participant is assigned to the median partner among all stable partners), whose existence had in fact been established earlier in Teo and Sethuraman [30] for the one-to-one stable matching problem. The existence of such a solution is not obvious: consider the assignment obtained by pairing each agent with his or her median stable partner; that such an assignment should be a matching is itself surprising, so it is quite amazing that the resulting solution is not only a matching, but is also stable!

In this paper, we establish a similar median solution concept for many-to-one (and the many-to-many) stable matchings. This is a useful complement to other studies on finding fair stable matchings, see [15]. More importantly, it proposes a new approach to address the asymmetry often observed in the stable matching problem by showing the existence of a solution that is fair in some sense to every participant involved. Moreover, this result as well as the earlier result in [30] can be viewed as a generalization of of an observation first made by Conway [14] on the lattice structure of stable matchings.

Several generalizations of the stable matching problem have been studied because of potential applications to NRMP and other labor markets. Of particular interest is the one in which certain pairs of students ("couples") would like to be assigned to hospitals that are geographically close together. In recent work, Cantala [8] considers a class of preferences in which (a) the "hospitals" are partitioned into regions; (b) any couple first decides on a ranking of the regions (on which they agree), and then ranks the hospitals within each region; their individual rankings of the hospitals within a given region may be different; and (c) their preferences are otherwise independent, and each of them only cares about the hospital they are assigned to. Even under this restricted class of preferences, Cantala [8] shows that a stable matching may not exist. We show that under this restricted class of preferences, one can determine efficiently whether or not a stable matching exists by simply solving a linear program. This gives rise to an alternate matching mechanism that can be used to produce "fair" stable admission solution when there are couples in the market, provided the concerns of the couples can be modeled by conditions (a)-(c), and that at least one stable solution exists. Note that the current NRMP matching mechanism may also fail to produce a stable matching in the case of couples.

The common theme underlying both our results is an understanding of the structure of "fractional" stable matchings. Baiou and Balinski [5] characterized the convex hull of all stable admissions solutions using linear inequalities in the natural assignment variables. Building on their results, we construct a geometric structure on the feasible fractional stable admission solutions and use this geometric structure to derive our main results. As a by-product, our approach gives a simple visual proof of the integrality of the stable admission polytope.

Organization of the paper. We introduce the stable admissions polytope in section 2. In section 3, we provide a geometric structure to the fractional stable admissions solutions. Several generalizations of some known results as well as simpler, compact formulations for this problem are discussed as well. Section 4 uses this structure to design a fair stable matching mechanism to the many-to-one stable admission problem. We also describe how the stable matching problem with couples can be addressed using this approach. We end with a brief summary.

2 The stable admissions model

An instance of the stable admissions problem consists of two sets of agents, the set $U = \{u_1, u_2, \ldots, u_{|U|}\}$ of "universities" and the set $A = \{a_1, a_2, \ldots, a_{|A|}\}$ of "students." Each agent has a strict, transitive, preference ordering over the agents on the other side of the market; in addition, associated with university u is a positive integer q_u representing its "quota," the interpretation being university u is allowed to admit up to q_u students. For ease of exposition, we assume that the preference orderings of the agents are complete, and that the number of students is exactly the total quota of the universities, i.e. $\sum_{u \in U} q_u = |A|$. We shall later argue that this assumption is without loss of generality.

A matching μ for a stable admissions problem is a subset of $U \times A$ such that university u appears in at most q_u pairs and student a appears at most once in μ. For convenience, we let $\mu(a)$ represent the university that student a is assigned to, and $\mu(u)$ represent the *set* of students assigned to university u. A matching μ is *stable* if there is no pair (u, a) such that student a prefers university $\mu(a)$ and university u prefers student a to at least one of its students in the set $\mu(u)$. By the classical result of Gale and Shapley [13], every instance of the stable admissions problem admits a stable matching. In fact, under our assumption of complete preferences and balance ($\sum_{u \in U} q_u = |A|$), every stable matching μ has *exactly* q_u pairs involving university u, and exactly one pair involving a student a.

The stable admissions problem with $q_u = 1$ for all $u \in U$ is called the *stable marriage* problem.

2.1 Definitions and Notation

The symbols $>_u$ and $>_a$ will indicate the preference orderings of university u and student a respectively; in particular, $a >_u b$ iff university u prefers student a to student b, and $a \geq_u b$ iff $a >_u b$ or $a = b$. Let $\Gamma = U \times A$, and

$$\Gamma^+ = \{(u,a) : |\{b : b >_u a\}| \geq q_u - 1\}.$$

In other words, a pair (u,a) belongs to Γ^+ if and only if university u prefers at least $(q_u - 1)$ students to student a. The *shaft*, $S(u,a)$, is defined for every $(u,a) \in \Gamma^+$ as

$$S(u,a) = \{(u,b) : b \geq_u a\}.$$

The *tooth*, $T(u,a)$, is defined for every $(u,a) \in \Gamma$ as

$$T(u,a) = \{(v,a) : v \geq_a u\}.$$

For any set of q_u students $\{a, a_1, \ldots, a_{q_u-1}\}$ with $a_i >_u a$ for all i, the *comb* $C(u, a, a_1, \ldots, a_{q_u-1})$ is the union of the shaft $S(u,a)$, the tooth $T(u,a)$, and the teeth $T(u,a_i)$, $i = 1, ..., q_u - 1$. Such a comb is said to be based at (u,a). Let C_u be the collection of all combs based at (u,a), for $a \in A$, and let $\mathcal{C} = \cup_{u \in U} C_u$.

To any stable admissions solution μ, we can associate a $|U| \times |A|$ incidence matrix x_μ (or x, if the stable matching μ is clear from the context) by letting $x_\mu(u,a) = 1$, if $(u,a) \in \mu$, and $x_\mu(u,a) = 0$ otherwise. For any $\hat{\Gamma} \subseteq \Gamma$, we let $x(\hat{\Gamma}) = \sum_{(u,a) \in \hat{\Gamma}} x(u,a)$.

The pair (u,a) blocks a matching μ if and only if $a >_u a'$ for some $a' \in \mu(u)$ and $u >_a \mu(a)$. In words, a pair (u,a) blocks μ if u prefers a to at least one of its assigned students in μ, and a prefers u to its assigned university in μ. We can express this condition using the notions of shaft and tooth defined earlier:

$$(u,a) \text{ blocks } \mu \text{ iff } x_\mu(S(u,a)) \leq q_u - 1 \text{ and } x_\mu(T(u,a)) = 0. \tag{1}$$

The convex hull of the incidence matrices of all stable admissions solutions is the stable admissions polytope (of that instance), discussed next.

2.2 The stable admissions polytope

Let $C(u, a, a_1, \ldots, a_{q_u-1})$ be a comb in C_u. Every stable admissions solution x must satisfy the *comb inequality*

$$x(C(u,a,a_1,\ldots,a_{q_u-1})) \equiv x(S(u,a)) + x(T(u,a) \setminus \{(u,a)\}) + \sum_{i=1}^{q_u-1} x(T(u,a_i) \setminus \{(u,a_i)\}) \geq q_u, \tag{2}$$

first described by Baiou and Balinski [5]. Otherwise, $x(S(u,a)) < q_u$, and $x(T(u,a')) = 0$ for some $a' \in \{a, a_1, a_2, \ldots, a_{q_u-1}\}$. As $a' \geq_u a$, $x(S(u,a')) \leq x(S(u,a)) \leq q_u - 1$; By (1), the pair (u,a') is a blocking pair in x. It is a simple matter to verify that for any unstable matching μ, one can find a violated comb inequality. These considerations show that the integer programming problem (see [5])

$$\sum_u x_{u,a} = 1, \quad \forall a \in A, \tag{3}$$

$$\sum_a x_{u,a} = q_u, \quad \forall u \in U, \tag{4}$$

$$x(C) = \sum_{(u,a)\in C} x_{u,a} \geq q_u, \quad \forall C \in C_u, u \in U. \tag{5}$$

$$x_{u,a} \in \{0,1\}, \qquad \forall (u,a) \in \Gamma \tag{6}$$

is an exact formulation of the stable admissions problem. Let P_{FSA} denote the set of all solutions to the associated (fractional) linear programming relaxation in which the constraints $x_{u,a} \in \{0,1\}$ are replaced by $x_{u,a} \geq 0$. Let P_{SA} be the convex hull of all stable matchings to the stable admissions problem. Clearly, $P_{SA} \subseteq P_{FSA}$. The main result of Baiou and Balinski [5] states that in fact $P_{SA} = P_{FSA}$.

In the next section we study the structure of fractional solutions of the stable admissions polytope; we exploit this structure in the following sections to derive a host of interesting results, including a simple proof of $P_{SA} = P_{FSA}$.

3 Geometry of fractional stable admissions

3.1 Decomposition

To motivate our main result, it is useful to think of each university u as owning q_u bins, each of size (= height) 1; each bin represents a "seat" and is indexed by the pair (i,u). Each $x_{ua} > 0$ will be treated as an "item" to be packed into one of the bins owned by university u. Bins are filled in the usual way (bottom-to-top); all items are packed in decreasing preference order of the university, following the procedure described next.

Phase t (for $t = 1, 2, \ldots$) of the procedure consists of (a) identifying the set, L_t, of bins with the maximum available space; and (b) assigning one item to each of the bins in L_t. The assignment of the items to the bins within a phase proceeds in a sequence of *steps*, indexed by $l = 1, 2, \ldots, |L_t|$. By our assumption, if bin $(i,u) \in L_t$ is considered in step l, university u's best remaining item is packed into it. We have not specified the order in which the bins are examined within each phase. As it turns out, the order does not matter.

At the end of any *step* of the packing procedure, let $S_{i,u}$ be the sequence of students assigned to seat i of university u; and let $a_{i,u}$ be university u's least preferred student in $S_{i,u}$. In terms of our bin-packing analogy, $a_{i,u}$ is the student occupying the top-most position in bin (i, u). Our main result is the following.

Theorem 1 *Consider any $x \in P^*$. Let $S_{i,u}$ be the sequence of students assigned to seat i of university u at the end of any step of the packing procedure. Let $a_{i,u}$ be university u's least preferred student in $S_{i,u}$. Then,*

(a) For all (i, u),

$$x(S_{i,u} \setminus a_{i,u}) + x(T(u, a_{i,u})) \;=\; x(S_{i,u}) + x(T(u, a_{i,u}) \setminus a_{i,u}) \;=\; 1;$$

(b) If item (v, a) is packed but (u, a) is not, then a prefers u to v.

(c) At the end of any phase, the $a_{i,u}$ are all distinct. In particular, for each $a \in A$, there is some (i, u) such that $a \equiv a_{i,u}$.

Before proving the theorem, it will be useful to consider an illustrative example.

3.2 Example

Consider the following example, adapted from Section 1.6.5 of Gusfield and Irving [14]. There are 5 universities and 11 students, with the preference lists and the quotas given in Tables 1 and 2.

There are a total of 7 stable matchings for this particular problem instance, and these are listed in Table 3.

Let x_{M_i} be the (integral) stable matching corresponding to M_i in the stable admission polytope, and let $x = (\sum_{i=1}^{7} x_{M_i})/7$ be a fractional solution in the stable matching polytope. We illustrate the packing procedure for this solution x.

Phase 1. Initially all the bins are empty, so L_1 consists of all the bins; each of these bins will have an item in it at the end of the first phase. Recall that items are packed by the universities in decreasing preference order.

a_1	u_3	u_1	u_5	u_4	
a_2	u_1	u_3	u_4	u_2	u_5
a_3	u_4	u_5	u_3	u_1	u_2
a_4	u_3	u_4	u_1	u_5	
a_5	u_1	u_4	u_2		
a_6	u_4	u_3	u_2	u_1	u_5
a_7	u_2	u_5	u_1	u_3	
a_8	u_1	u_3	u_2	u_5	u_4
a_9	u_4	u_1	u_5		
a_{10}	u_3	u_1	u_5	u_2	u_4
a_{11}	u_5	u_4	u_1	u_3	u_2

Table 1: Preference lists for the students

capacity												
(4)	u_1	a_3	a_7	a_9	a_{11}	a_5	a_4	a_{10}	a_8	a_6	a_1	a_2
(1)	u_2	a_5	a_7	a_{10}	a_6	a_8	a_2	a_3	a_{11}			
(3)	u_3	a_{11}	a_6	a_8	a_3	a_2	a_4	a_7	a_1	a_{10}		
(2)	u_4	a_{10}	a_1	a_2	a_{11}	a_4	a_9	a_5	a_3	a_6	a_8	
(1)	u_5	a_2	a_4	a_{10}	a_7	a_6	a_1	a_8	a_3	a_{11}	a_9	

Table 2: Quotas and preference lists for the universities

Matching	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
M_1	u_3	u_1	u_4	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_5
M_2	u_1	u_3	u_4	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_5
M_3	u_3	u_1	u_5	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_4
M_4	u_1	u_3	u_5	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_4
M_5	u_5	u_3	u_3	u_4	u_1	u_3	u_2	u_1	u_1	u_1	u_4
M_6	u_5	u_4	u_3	u_1	u_1	u_3	u_2	u_3	u_1	u_1	u_4
M_7	u_4	u_4	u_3	u_1	u_1	u_3	u_2	u_3	u_1	u_5	u_1

Table 3: List of all Stable Matchings

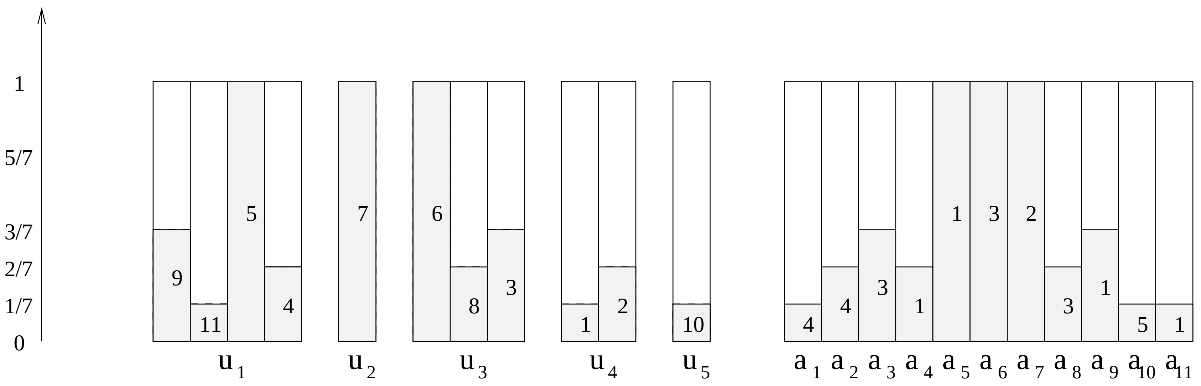

Figure 1: Bins at the end of Phase 1

Consider university u_1, which owns four bins. Its four best students in x are a_9, a_{11}, a_5, and a_4, with the corresponding "values" being 3/7, 1/7, 1, and 2/7 respectively. Thus, the four bins owned by u_1 contain these four items as shown in Figure 1. For convenience, we may also assume that also owns a bin in which items are packed: whenever item (u, a) is packed into bin (i, u), we may also assume that item (u, a) is packed into the bin owned by student a. Thus, students a_9, a_{11}, a_5, and a_4 each pack an item into their bins. In terms of this picture, it is easy to interpret that the three statements of Theorem 1: part (b) simply says that this packing, when viewed from the point of view of the students, proceeds in increasing preference order; given this, part (a) says that the occupied space in any bin with student a on top plus the available space in student a's bin is exactly 1; and part (c) says that at the end of any phase, the students on top of the bins owned by the universities are all distinct. All of these properties are easily verified from Figure 1, which shows the state of the bins at the end of phase 1.

Phase 2. The set L_2 will consist of all the bins that have the maximum available space; from Figure 1, it is clear that L_2 consists of bin 2 of u_1, bin 1 of u_4 and the bin owned by u_5. During phase 2, each of these three bins will have an item assigned to it. In fact, the top students in these bins will rearrange themselves, so that part (c) continues to hold. University u_1 will pack its most preferred remaining items, which happens to be student a_{10} (value 6/7); similarly, universities u_4 and u_5 will pack respectively items involving students a_{11} (value 4/7) and a_1 (value 2/7) respectively. The state of the bins at the end of phase 2 is shown in Figure 2; it is easy to verify that parts (a)-(c) of Theorem 1 continue to hold.

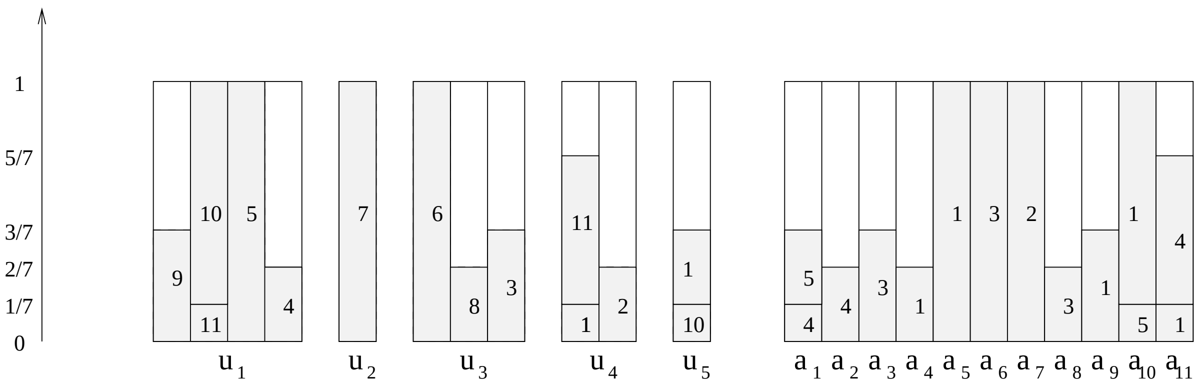

Figure 2: Bins at the end of Phase 2

Phase 3. Now L_3 consists of bin 4 of u_1, bin 2 of u_3 and bin 2 of u_4. The respective universities pack their most preferred remaining item into these bins, resulting in Figure 3.

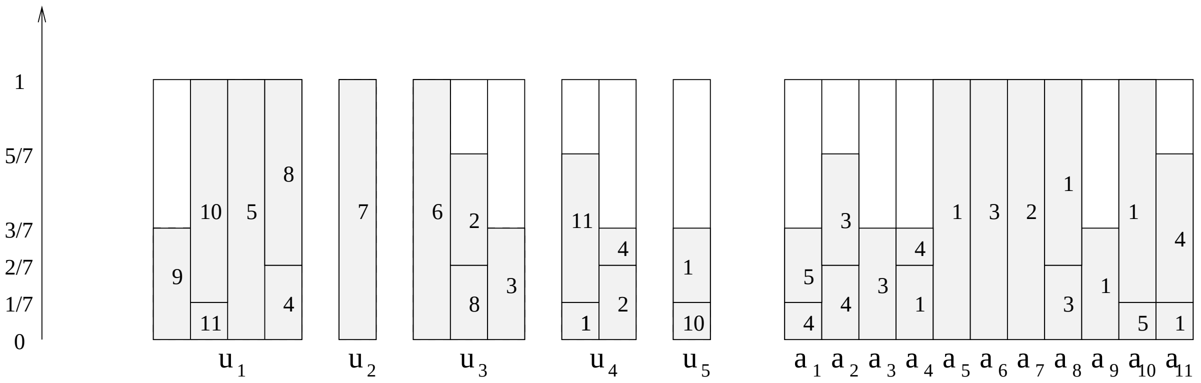

Figure 3: Bins at the end of Phase 3

Phase 4. The set L_4 consists of bin 1 of u_1, bin 3 of u_3, bin 2 of u_4, and the bin of u_5; each of these bins picks up an extra item, chosen to be the corresponding university's best remaining item. The state of the bins at the end of this phase appears in Figure 4.

Phase 5. The set L_5 consists of bin 1 of u_1, bin 2 of u_3, bin 1 of u_4, and the bin of u_5. Each of these bins is packed with an item, and now all the items are packed, which completes our packing procedure. The final state of the bins is in Figure 5.

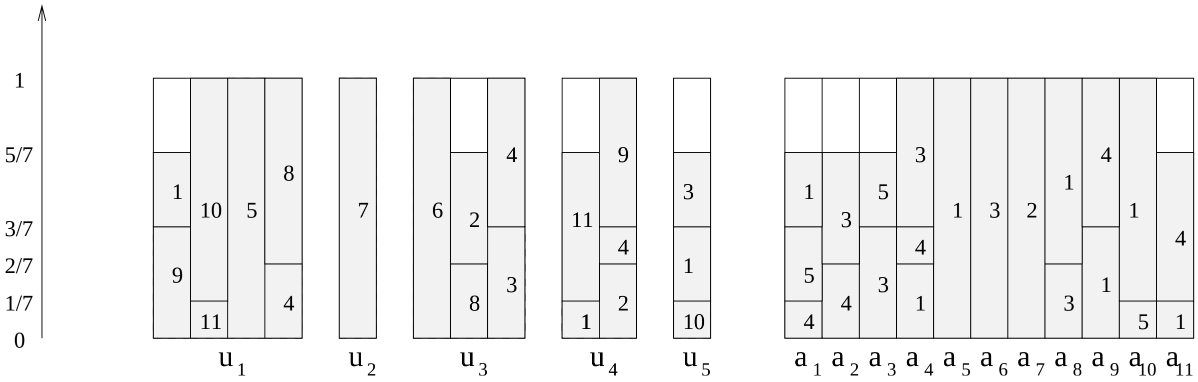

Figure 4: Bins at the end of Phase 4

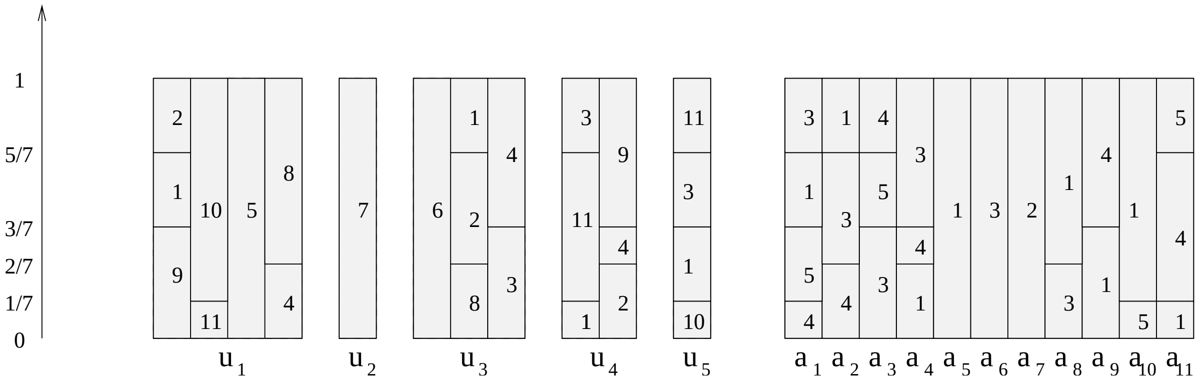

Figure 5: Bins at the end of Phase 5

3.3 Proof of Theorem 1

Our proof proceeds by induction on t. First, consider the case $t = 0$. Since we start with a set of empty bins, all of the bins will be in L_0; at the end of the phase, it is clear that university u will have (fractionally) packed its q_u most preferred items, one in each of its bins. For university u, let $S_u = \{a : x_{ua} > 0\}$, and let $B_u(x)$ be the q_u most preferred students (according to u) in S_u. (Note that S_u must have at least q_u students.) Consider the comb $C(u, B_u(x))$. Note that

$$S(u,a) = B_u(x)$$

if a is the least preferred student in $B_u(x)$, according to university u. Clearly,

$$x(C(u, B_u(x))) \;=\; \sum_{a \in B_u(x)} x(T(u,a)).$$

By the comb inequality,

$$x(C(u, B_u(x))) \;\geq\; q_u,$$

whereas by the student assignment constraint,

$$\sum_{a \in B_u(x)} x(T(u,a)) \;\leq\; q_u.$$

Therefore, $x(T(u,a)) \;=\; 1$ for each $a \in B_u(x)$; this, combined with the observation that $S_{i,u} = a_{i,u}$, proves part (a). Part (b) is true because $x(T(u, a_{i,u})) = 1$, and $x_{u,a_{i,u}} > 0$ for all (i,u), and preferences are strict. Moreover, if $a_{i,u} = a_{j,v} = a$, then either $x(T(u,a)) < 1$ or $x(T(v,a)) < 1$, depending on whether $v >_a u$ or $u >_a v$; so, the $a_{i,u}$ are all distinct at the end of phase 0. Since the number of students is the same as the total number of bins, part (c) follows.

Suppose the Theorem is true up to the end of phase t. We now establish the Theorem for phase $t+1$. Let α be the maximum space available in any bin during phase $t+1$. Consider any item (u,a) that has not yet been packed. From part (c), $a \equiv a_{j,v}$ for some bin (j,v); from parts (a) and (b), we have

$$x(T(u,a)) \;\leq\; x(T(v,a)) - x_{v,a} \;=\; 1 - x(S_{j,v}) \;\leq\; \alpha, \tag{7}$$

where the last inequality follows because each bin has at most α available space. Moreover,

$$x(T(u,a)) \;=\; \alpha \quad \text{implies} \quad (j,v) \in L_{t+1}. \tag{8}$$

Suppose $(i,u) \in L_{t+1}$ is chosen in any step of phase $t+1$, and item x_{ua} is to be packed into bin (i,u) next. Consider the comb

$$C_{u,a} \equiv C(u, a, a_{1,u}, a_{2,u}, \ldots, a_{i-1,u}, a_{i+1,u}, \ldots, a_{q_u,u}),$$

which contains

- the teeth $T(u,a)$, $T(u, a_{j,u})$ for $j = 1, 2, \ldots, i-1, i+1, \ldots q_u$; and
- the shaft $S(u,a) = \{(u,a)\} \cup S_{1,u} \cup S_{2,u} \cup \ldots \cup S_{q_u,u}$.

By the comb inequality, we have

$$q_u \;\leq\; x(C_{u,a}) \;=\; \sum_{j:j\neq i} x(T(u,a_{j,u})) + x(T(u,a)) + x(S_{i,u}) + \sum_{j:j\neq i} x(S_{j,u} \setminus \{a_{j,u}\}).$$

By induction, for $j \neq i$,

$$x(T(u,a_{j,u})) + x(S_{j,u} \setminus \{a_{j,u}\}) \;=\; 1,$$

which reduces the comb inequality to

$$x(T(u,a)) + x(S_{i,u}) \;\geq\; 1. \tag{9}$$

Since $(i,u) \in L_{t+1}$ and has not been packed earlier, $x(S_{i,u}) = 1 - \alpha$; from Eq. (7), $x(T(u,a)) \leq \alpha$. So the (reduced) comb inequality (9) must be satisfied as an equality. We next pack (u,a) in bin (i,u), and update $S_{i,u}$ and $a_{i,u}$, and observe that the reduced comb equality can be written as

$$x(T(u,a_{i,u})) + x(S_{i,u} \setminus a_{i,u}) \;=\; 1,$$

establishing part (a). Part (b) follows since if item (u,a) is not packed but item (v,a) is packed, then from Eq. (7),

$$x(T(u,a)) \leq \alpha \leq x(T(v,a)).$$

Hence a prefers u to v.

Observe that $x(T(u,a)) = \alpha$ for every item (u,a) packed in phase $t+1$, and these are the least preferred items in their respective bins; this implies the least preferred items in the newly packed bins are all distinct. From (8), if a bin (i,u) was not selected in phase $t+1$, its least preferred student could not have been part of any item packed in phase $t+1$. So the $\{a_{i,u}\}$ remain distinct at the end of phase $t+1$. ∎

3.4 Consequences

Let $x \in P^*$. For any $\alpha \in (0,1]$, let

$$I_u^\alpha(x) = \{a : 1 - x(T(u,a)) \le \alpha < 1 - x(T(u,a)) + x_{ua}\} \tag{10}$$

and $I^\alpha(x) = \cup_{u \in U} I_u^\alpha(x)$. By part (a) of Theorem 1, $I_u^\alpha(x)$ consists of those students whose assignment to their "seat" in university u changes the occupied space in that seat from at most α to some quantity strictly less than α. Since each university fills all its seats, the following corollary of Theorem 1 is immediate.

Corollary 1 *For any $x \in P^*$ and any $\alpha \in (0,1]$, $|I_u^\alpha(x)| = q_u$.*

Remark. Baiou and Balinski [5] observed that $I_u^0(x) = I_u^1(x) = q_u$ for any stable admissions solution x; Corollary 1 generalizes these observations. Even though we have excluded $\alpha = 0$ in the statement of the corollary, it is clear that $I_u^0(x) = I_u^\epsilon(x)$ for a sufficiently small value of $\epsilon > 0$.

Roth, Rothblum and Vande Vate [26] proved the following result for the stable *marriage* polytope using a primal-dual approach.

Lemma 2 *Let $q_u = 1$ for all $u \in U$. For any $x \in P^*$, $x_{u,a} > 0$ implies*

$$x_{u,a} + x(S(u,a) \setminus (u,a)) + x(T(u,a) \setminus (u,a)) \;=\; 1.$$

This result was crucial in proving that P^* is exactly the convex hull of all stable marriage solutions [26, 30]. The following result generalizes Lemma 2 to the stable admissions setting; its proof is implicit in the proof of Theorem 1 and is therefore omitted.

Lemma 3 *For any $x \in P^*$, let $C_{\alpha,u}$ denote the comb obtained by using the q_u students in $I_u^\alpha(x)$ as the bases for the teeth in the comb. Then, $x(C_{\alpha,u}) = q_u$.*

We next show how the main result of Baiou and Balinski [5] follows as an immediate consequence of Theorem 1.

Theorem 4 *The inequalities (3)-(6) determine the stable admissions polytope P_{SA}.*

Proof. Given any $x \in P^*$, generate a number $\alpha \in (0,1]$ uniformly at random; for each university u, assign the q_u students in $I_u^\alpha(x)$ to university u to get an assignment μ_α. By part (c) of Theorem 1, no student is assigned to two different universities under μ_α, so that μ_α is a matching. Consider any student b whom u strictly prefers to at least one of its assigned students (say a) under μ_α; let $v = \mu_\alpha(b)$. The item (u,b) would have been packed before item (u,a) by the packing procedure; moreover, at the end of the phase, say t, in which (u,b) was packed, the space available in the seat occupied by student b at university u must be strictly greater than $(1-\alpha)$, otherwise u cannot prefer b to a. Thus, at the end of phase t, (u,b) was packed, but (v,b) was not, and so by part (b) of Theorem 1, b prefers v to u. This implies μ_α is stable. By construction, the expected value of $\mu_\alpha(u,a)$ is $x_{u,a}$. Hence any $x \in P^*$ can be written as $E(\mu_\alpha)$, i.e., convex combination of stable matching solutions. Integrality of the polytope is now immediate [30]. ■

The following corollary is yet another application of the structural property; the special case of $\alpha = 0, \beta = 1$ was proved by Baiou and Balinski [5].

Corollary 2 *For $x \in P^*$, $x \in (0,1)$, and any $\alpha, \beta \in (0,1]$, define*

$$x_{ua}^{\alpha,\beta,\epsilon} = \begin{cases} x_{ua} + \epsilon & \text{if } (u,a) \in I^\alpha(x) \text{ and } (u,a) \notin I^\beta(x), \\ x_{ua} - \epsilon & \text{if } (u,a) \in I^\beta(x) \text{ and } (u,a) \notin I^\alpha(x), \\ x_{ua} & \text{otherwise}, \end{cases}$$

for $|\epsilon| > 0$ arbitrarily small. Then $x^{\alpha,\beta,\epsilon} \in P^*$.

Proof. Let X_u be the assignment obtained in the proof of Theorem 4, where u is the number generated using a uniform random distribution on $(0,1]$. The proof of Theorem 4 implies that x can be written as a convex combination of X_u as u varies over the interval $(0,1]$. Note that

$$x^{\alpha,\beta,\epsilon} = x + \epsilon\Big(X_\alpha - X_\beta\Big).$$

It is thus easy to verify that $x^{\alpha,\beta,\epsilon}$ can be also written as convex combination of X_u.

Therefore $x^{\alpha,\beta,\epsilon} \in P_{SA}$. ■

3.5 A compact formulation

In the proof of Theorem 1, we showed how all the fractional partners of any university u in a solution $x \in P^*$ can be partitioned into q_u sets. The essential result here is that the given fractional stable solution x can be written as a convex combination of (integral) stable matchings;

in each stable matching participating in this convex combination, university u will be matched to exactly one student from each of these q_u sets.

Now we show that one can in fact construct a *single* partition of the students that works for *all* fractional solutions. The advantage of such a construction is clear: all of the results on the stable admissions problem can thus be reduced to that of the stable marriage problem by appealing to this transformation. Indeed, given this decomposition, it is a simple matter to find a compact linear formulation to express the convex hull of all stable matchings. The disadvantage is that constructing such a decomposition entails knowing the entire set of stable assignments for each university u. The results of this subsection have been independently derived by Fleiner [12] using similar methods.

The motivation for our result is Theorem 1.6.4 (page 46) of Gusfield and Irving [14], reproduced below (with the obvious modification in terminology); this was first proved by Roth and Sotomayor [25].

Theorem 5 *Given 2 stable matchings M and M', suppose university u is assigned non-identical sets of students in M and M'. If u prefers its least favored student in $M \setminus M'$ to its least favored student in $M' \setminus M$, then, u prefers all the students in M to any student in $M' \setminus M$. (here we say u likes M better than M')*

For sake of completeness, we derive a simple proof to the above using Theorem 1.

Proof. Let x_M and $x_{M'}$ denote the (integral) stable admission solution corresponding to the matching M and M' respectively.

Let $x = (x_M + x_{M'})/2$, and consider the packing process using the fractional solution x. For university u, the q_u most favored items, denoted by Q_u, will be packed first at the initial stage. Note that if student a is assigned to u in $M \cap M'$, then $x_{ua} = 1$. Hence (u, a) must be among the q_u most preferred items packed at the initial phase, otherwise condition (a) in Theorem 1 will be violated.

According to university u, let s and s' be the least favored student in $M \setminus M'$ and $M' \setminus M$ respectively. By assumption, u prefers s to s'.

Suppose the contrary, and u prefers a student in $M' \setminus M$ to some students in M. s cannot be one of the q_u most favored items. From condition (c) of Theorem 1, the student s must be assigned to some university u' (under M'), and s is one of the $q_{u'}$ most preferred items according to u'. Since $x(T(u', s)) = 1$, s prefers u to u'. Now, since u prefers s to s', (u, s) forms a blocking pair for the stable assignment M'. This is a contradiction. ■

The key conclusion emerging from Theorem 5 is that any university either prefers all of its M-partners to any of its M'-partners who are not also M-partners, or prefers all of its M'-partners to any of its M-partners who are not also M'-partners. In the former case, we say that university u prefers M to M', and in the latter, that it prefers M' to M. In particular, there cannot be two *different* stable assignments for university u which agree on its worst student. Thus, each university can rank order all of its stable assignments in decreasing order of its worst student. All possible stable assignments for a university u can be determined in polynomial time by an iterative application of the Gale-Shapley algorithm. Let $\mu_1, \mu_2, \ldots, \mu_K$ be the ordered list of stable assignments for university u, with

$$\mu_1 >_u \mu_2 >_u \ldots >_u \mu_K. \tag{11}$$

Note that all the μ_i have the same cardinality; in fact, by our assumption, $|\mu_i| = q_u$ for each i.

Theorem 6 *The set of all students matched to university u in any stable matching can be partitioned into sets $S_i(u)$, $i = 1, 2, \ldots, q_u$, so that each stable matching contains exactly one student from each $S_i(u)$.*

Proof. We shall now describe a procedure to construct the decomposition; to emphasize the parallel between this and the earlier proof of Theorem 1, we shall use the terminology of "bins" and "items." As in the earlier proof, assign each item of μ_1 to a distinct bin. Now, consider the set $S = \mu_2 \setminus \mu_1$ and $T = \mu_1 \setminus \mu_2$. Since μ_1 and μ_2 are distinct, and have the same cardinality, S and T are *non-empty* and have the same cardinality; each item of T already occupies a distinct bin, so each item of S can be assigned to some bin occupying an item of T. Now, stable assignments μ_1 and μ_2 have at most one item from each bin.

Next, look at the set $S = \mu_3 \setminus (\mu_1 \cup \mu_2)$; we claim that $S = \mu_3 \setminus \mu_2$. For otherwise, there must be a student a such that $a \in \mu_1$, $a \in \mu_3$, but $a \notin \mu_2$. Since u prefers μ_1 to μ_2, u must prefer a to any student in $a' \in \mu_2 \setminus \mu_1$; and since u prefers μ_2 to μ_3, u must prefer a' to any student in $\mu_3 \setminus \mu_2$. In particular, u must prefer a' to a, which is a contradiction. Thus, $S = \mu_3 \setminus \mu_2$. Now, consider $T = \mu_2 \setminus \mu_3$. Again, both S and T are non-empty and have the same cardinality, every item in T already appears in a distinct bin, so each item in S can also be assigned to a bin containing an item of T. Now, each of μ_1, μ_2 and μ_3 contains exactly one item from each bin. (This is because only the items in S are newly assigned, none of which can be present in $\mu_1 \cup \mu_2$.)

We are now ready to do an inductive proof: at any stage i, let $S = \mu_i \setminus \mu_{i-1}$, $T = \mu_{i-1} \setminus \mu_i$; clearly,

$$S = \mu_i \setminus \{\mu_1 \cup \mu_2 \cup \ldots \cup \mu_{i-1}\}.$$

Also, S and T are non-empty, have the same cardinality; all items in T are in distinct bins, so the items in S can be assigned to distinct bins, each containing a distinct item in T. Moreover, we have now ensured that the stable assignments $\mu_1, \mu_2, \ldots \mu_i$ all contain exactly one item from each bin; this is because only the items in S are newly assigned, none of which can be present in any of the assignments $\mu_1, \mu_2, \ldots, \mu_{i-1}$. ■

Note that Theorem 6 helps us reduce the stable admissions problem to a "stable marriage" like problem: use q_u "men" instead of university u, with man (i, u) declaring only the students in $S_i(u)$ as acceptable.

Remark. This decomposition extends to the many-to-many case, as Theorem 1.6.4 can be extended; see Balinski and Baiou [4]. Specifically, given any fractional many-to-many stable matching, we can do the usual construction by letting universities "pack" their agents best to worst in each of their "seats," from left to right; and similarly let agents "pack" their universities best to worst, starting from the right. So the interesting question that remains is: is there an analog of the comb formulation for the many-to-many case?

Finally, we argue that our assumption of complete preferences and balance (total quota of the universities = total number of students) is without loss of generality. Let $q_1, q_2, \ldots, q_m$ be the quotas of the m universities and let there be n students. Let $Q = \sum_i q_i$. We introduce Q dummy students, with q_i students corresponding to university i; moreover we introduce n dummy universities, each with a quota of one, and corresponding to each real student. The rankings of the universities and the students are as follows: each university rank orders real students, followed by the q_i dummy students corresponding to it in some order; each real student j appends the corresponding dummy university j to his list; each dummy university ranks the corresponding real student first, followed by all dummy students in some order; finally, a dummy student ranks her corresponding real university first, followed by dummy universities in arbitrary order. It is easy to verify that any stable matching in the original instance can be extended to a stable matching in the new instance by filling unused real slots with dummy students, assigning unmatched real students to dummy universities, and matching the remaining dummy students to dummy universities. Similarly, any stable matching in this new instance can be shown to correspond to a stable matching in the old instance. Finally, even though the preferences are incomplete in this new instance, no agent can be unmatched in any stable matching. Thus, appending the "missing" agents on the other side to each preference list in an arbitrary way cannot change the set of stable matchings.

4 Applications

In the rest of this section, we use the geometry of fractional solutions of the stable admissions problem to address two natural applications to this problem: the design of *fair* solutions, and the problem of *college admissions for couples.*

4.1 A Fair Solution Concept

The proposal algorithm due to Gale and Shapley [13] can be used to find a stable matching in the stable admissions problem. Unfortunately, this algorithm will generate one of two extremal solutions: the university-optimal or the student-optimal stable solution. It is well known that the university-optimal stable solution is the worst solution for the students; and that the student-optimal solution is the worst solution for the universities.

Suppose there are exactly N stable matchings in a stable admission problem. Let $x_1, \ldots, x_N$ be the stable solutions to this problem. The challenge here is to choose one of these solutions to produce a stable matching of all the participants in the stable admission problem. Is there a way to choose a solution where all participants will be fairly treated?

We define a point x_f in the convex hull P_{SA} by taking

$$x_f(u,a) = \frac{\sum_{j=1}^{N} x_j(u,a)}{N}$$

for all (u,a). We have now a fairly complete understanding of the geometric structure of this fractional solution, using the above packing procedure (cf.Figure 5). The geometric structure of x_f is extremely useful, as we can now immediate read off many possible stable admission solutions from the layout.

- University Optimal Solution: Match u to the q_u students in $I_{0^+}(u)$. The solution obtained is university-optimal (best q_u possible students are assigned) but student-pessimal (worst stable university assigned).

In Figure 5, this corresponds to choosing all items touching the bottom of every bins in the figure, i.e., touching the line $\alpha = 0$. Note that by our construction, all bins corresponding to universities will be assigned their best possible set of students (with preference indicated by (11)). All the students will be assigned their worst possible stable partners.

- Student Optimal Solution: Match u to the q_u students in $I_1(u)$. The students are assigned to their best possible stable university (i.e., student-optimal), and the universities are assigned to their worst possible set of students.

  In Figure 5, this corresponds to choosing all items touching the top of every bins in the figure, i.e., touching the line $\alpha = 1$. Note that by our construction, all bins corresponding to universities will be assigned their worst possible set of students among all stable solutions (with preference indicated by (11)). All the students will be assigned their best possible stable partners.

- If N is odd, we can define the *median* stable admission solution as follows: Match u to the q_u students in $I_{0.5}(u)$. The students are thus matched to their median stable university (counting multiplicity). By the packing procedure of section 3.1 and the discussion in section 3.5, it is clear that each university is also matched with its median set of students.

  In Figure 5, this corresponds to choosing all items crossed by the line $\alpha = 0.5$, i.e., occupying the halfway mark of every bin. The matching obtained: $u_1 \leftarrow \{1, 5, 8, 10\}$, $u_2 \leftarrow \{7\}$; $u_3 \leftarrow \{2, 4, 6\}$; $u_4 \leftarrow \{9, 11\}$; $u_5 \leftarrow \{3\}$. This corresponds to the solution M_4, which is stable. M_4 is thus the median stable solution.

This discussion can be summarized as the following theorem.

Theorem 7 *There exists a stable solution to the college admission problem with the following properties:*

- *Assign each student to the median university, among all universities assigned to in the stable solutions;*
- *Assign each university its median set of students, among all stable admission solutions, and where the set of students assigned are ranked ordered based on the ranking of the students in lexicographical manner (cf. (11)).*

The existence of a matching mechanism that assigns all participants to their median set of partners is indeed surprising. One of the limitation of this mechanism though is that all the stable matching solution must be enumerated before a median solution can be constructed. However, since computational time is not a concern in this class of problem, all the stable solutions can theoretically be obtained by enumerating all the corner points of the polytope given by the LP characterization of the stable admission polytope.

4.2 Stable admissions with couples

Finally, we consider the stable admissions problem with couples. Many of the appealing mathematical results for the stable matching program do not carry over to the case where couples express preferences over pairs of positions. For example it is observed that the following results are valid [2].

(i) There are instances of the couples problem that do not have a stable matching;

(ii) Even if an instance of the couples problem has a stable matching, it may not have a hospital optimal or student optimal stable matching;

(iii) In an instance of the couples problem in which the preference lists are not complete, there may be stable matchings which leave different numbers of positions unfilled.

In this section we consider a special case of the stable admissions problem with couples with a special preference structure. Specifically, we assume that the hospitals are partitioned into several groups based on their location. (For example, group together all of the hospitals located in the same city.) Each husband and wife submit separate rankings of the universities subject to the following restriction:

(a) The ranking is first done by ranking the city and then ranking the universities;

(b) If a person prefers group 1 to group 2, then all universities in group 1 will be ranked above any university in group 2; and

(c) Each couple must rank the groups in the same way, but are allowed to submit different rankings of the universities within a group.

The essential geographic preferences of the couples are captured by allowing the husband and wife to be assigned to different hospitals as long as these hospitals are within the same group (i.e., in the same region). Thus, we wish to find a stable matching in which each couple is assigned to hospitals within the same group. Cantala [8] considers a related model and shows that a stable matching may not exist even in this restricted setting. Here we show that a variation of the linear inequality description for the stable admission polytope remains valid for this version of the stable matching problem with couples; in particular, a stable solution, if it exists, can be found using linear programming.

We claim that the following inequalities give the polytope of stable admission with couples (P_{SAC}):

$$x_{u,a} \geq 0, \quad \forall (u,a) \in \Gamma, \tag{12}$$

$$\sum_u x_{u,a} = 1, \quad \forall a \in A, \tag{13}$$

$$\sum_a x_{u,a} = q_u, \quad \forall u \in U, \tag{14}$$

$$\sum_{j \in K} x_{p,j} = \sum_{j \in K} x_{q,j}, \quad \text{for all couple } (p,q) \text{ and group } K, \tag{15}$$

$$x(C) = \sum_{(u,a) \in C} x_{u,a} \geq q_u, \quad \forall C \in C_u, u \in U. \tag{16}$$

Here the inequalities are called the non-negativity, student, university, couple and comb constraints respectively.

The fractional solution of (P_{SAC}) still satisfies the decomposition property of section 3.1, since it is a subset of the stable admission polytope. Again, using the randomized rounding approach, the integrality property is observed. Note that by our rounding method, the couple constraints are automatically satisfied for the $0-1$ solution obtained by the rounding method: If a person is assigned to a hospital in region K, the spouse will also be assigned to a hospital in the same region, because of the constraint

$$\sum_{j \in K} x_{p,j} = \sum_{j \in K} x_{q,j}, \quad \text{for all couple } (p,q) \text{ and group } K.$$

Theorem 8 *The convex hull of the stable admission solutions to the couples problem satisfying conditions (a), (b) and (c) is given by (P_{SAC}).*

Using the same argument in our derivation of the median solution concept, and the fact that $(P_{SAC}) \subset (P_{SA})$, we have the next result:

Theorem 9 *If $(P_{SAC}) \neq \emptyset$, then a median solution exists for the stable admission problem with couples.*

5 Conclusion

In this paper, we study the stable admissions problem in a geometric approach. We found the decomposition property of the admissions polytope, which allows us to express fractional solutions in the stable admissions polytope as convex combinations of integral solutions. The results provides new insights into the stable admissions problem, resulting in the generalization of many structural properties of the stable marriage polytope to this setting.

A challenging problem is to consider the stable marriage polytope with indifferences. The stable admissions problem can be viewed as a special case. However, since the problem of determining whether a given pair is stable is NP-complete for a given instance of stable marriage with ties [19], this appears to be a difficult problem.

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