# Second-Order Cone Programming

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February 18, 2002

## 1 Introduction

Second-order cone programming (SOCP) problems are convex optimization problems in which a linear function is minimized over the intersection of an affine linear manifold with the Cartesian product of second-order (Lorentz) cones. Linear programs, convex quadratic programs and quadratically constrained convex quadratic programs can all be formulated as SOCP problems, as can many other problems that do not fall into these three categories. These latter problems model applications from a broad range of fields from engineering, control and finance to robust optimization and combinatorial optimization.

On the other hand semidefinite programming (SDP)—that is the optimization problem over the intersection of an affine set and the cone of positive semidefinite matrices—includes SOCP as a special case. Therefore, SOCP falls between linear (LP) and quadratic (QP) programming and SDP. Like LP, QP and SDP problems, SOCP problems can be solved in polynomial time by interior point methods. The computational effort per iteration required by these methods to solve SOCP problems is greater than that required to solve LP and QP problems but less than that required to solve SDP's of similar size and structure. Because the set of feasible solutions for an SOCP problem is not polyhedral as it is for LP and QP problems, it is not readily apparent how to develop a simplex or simplex-like method for SOCP.

While SOCP problems can be solved as SDP problems, doing so is not advisable both on numerical grounds and computational complexity concerns. For instance, many of the problems presented in the survey paper of Vandenberghe and Boyd [VB96] as examples of SDPs can in fact be formulated as SOCPs and should be solved as such. In §2, 3 below we give SOCP formulations for four of these examples: the convex quadratically constrained quadratic programming (QCQP) problem, problems involving fractional quadratic functions

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such as those that arise in structural optimization, logarithmic Tchebychev approximation and the problem of finding the smallest ball containing a given set of ellipsoids. Thus, because of its broad applicability and its computational tractability, SOCP deserves to be studied in its own right.

Particular examples of SOCP problems have been studied for a long time. The classical Fermat-Weber problem (described in §2.2 below) goes back several centuries; see [Wit64] and more recently [XY97] for further references and background. The paper of Lobo et al [LVBL98] contains many applications of SOCP in engineering. Nesterov and Nemirovski [NN94], Nemirovski [Nem99] and Lobo et al [LVBL98] show that many kinds of problems can be formulated as SOCPs. Our presentation in §2 is based in part on these references. Convex quadratically constrained programming (QCQP), which is a special case of SOCP, has also been widely studied in the last decade. Extension of Karmarkar's interior point method [Kar84] to QCQPs began with Goldfarb, Liu and Wang [GLW91], Jarre [Jar91] and Mehrotra and Sun [MS91].

Nesterov and Nemirovski [NN94] showed that their general results on self-concordant barriers apply to SOCP problems, yielding interior point algorithms for SOCP with an iteration complexity of $\sqrt{r}$ for problems with $r$ second-order cone inequalities (see below for definitions). Nemirovski and Scheinberg [NS96] showed that primal or dual interior point methods developed for linear programming can be extended in a word-for-word fashion to SOCP; specifically they showed this for Karmarkar's original method.

Next came the study of primal-dual interior point methods for SOCP. As in linear and semidefinite programming, primal-dual methods seem to be numerically more robust for solving SOCPs. Furthermore, exploration of these methods leads to a large class of algorithms, the study of which is both challenging and potentially significant in practice. Study of primal-dual interior point methods for SOCP started with Nesterov and Todd [NT97, NT98]. These authors presented their results in the context of optimization over self-scaled cones, which includes the class of second-order cones as special case. Their work culminated in the development of a particular primal-dual method called the NT method. Adler and Alizadeh [AA95] studied the relationship between semidefinite and second-order cone programs and specialized the so-called $XZ + ZX$ method of [AHO98] to SOCP. Then Alizadeh and Schmieta [AS97] gave nondegeneracy conditions for SOCP and developed a numerically stable implementation of the $XZ+ZX$ method; this implementation is used in the SDPPACK software package, [AHN+97]. (§5 and §6 below are partly based on [AS97].) Subsequently, Monteiro and Tsuchiya in [MT00] proved that this method, and hence all members of the Monteiro-Zhang family of methods, have a polynomial iteration complexity.

There is a unifying theory based on Euclidean Jordan algebras that connects LP, SDP and SOCP. The text of Faraut and Korány [FK94] covers the foundations of this theory. We review this theory for the particular Jordan algebra relevant to SOCP problems in §4. Faybusovich [Fay97b, Fay97a, Fay98] studied nondegeneracy conditions and presented the NT, $XZ$ and $ZX$ methods in Jordan algebraic terms. Schmieta [Sch99] and Schmieta and Alizadeh [SA01, SA99] extended the analysis of the Monteiro-Zhang family of interior point algorithms from SDP to all symmetric cones using Jordan algebraic techniques. They also showed that word-for-word generalizations of primal based and dual based interior point methods carry over to all symmetric cones [AS00]. In addition, Tsuchiya [Tsu97, Tsu99] used Jordan algebraic techniques to analyze interior point methods for SOCP. The overview of the path-following methods in §7 is partly based on these references.

There are now several software packages available that can handle SOCPs or mixed SOCP, LP and SDP problems. The SDPpack package was noted above. Sturm's SeDuMi [Stu98] is another widely available package that is based on the Nesterov-Todd method; in [Stu99] Sturm presents a theoretical basis for his computational work.

In this paper, we present an overview of the SOCP problem. After introducing a standard form for SOCP problems and some notation and basic definitions, we show in §2 that LP, QP, quadratically constrained QP, and other classes of optimization problems can be formulated as SOCPs. We also demonstrate how to transform many kinds of constraints into second-order cone inequalities. In §3 we describe how robust least squares and robust linear programming problems can be formulated as SOCPs. In §4 we describe the algebraic foundation of second-order cones. It turns out that a particular Euclidean Jordan algebra underlies the analysis of interior point algorithms for SOCP. Understanding of this algebra helps us see the relationship between SDP and SOCP more clearly.

Duality and complementary slackness for SOCP are covered in §5 and notions of primal and dual non-degeneracy and strict complementarity are covered in §6. In §7, the logarithmic barrier function and the equations defining the central path for an SOCP are presented and primal-dual path-following interior point methods for SOCPs are briefly discussed. Finally in §8 we deal with the efficient and numerically stable implementation of interior point methods for SOCP.

## Notation and Definitions

In this paper we work primarily with optimization problems with block structured variables. Each block of such an optimization problem is a vector constrained to be inside a particular second-order cone. Each block is a vector indexed from 0. We use lower case boldface letters $\mathbf{x}$, $\mathbf{c}$ etc. for column vectors, and uppercase letters $A$, $X$ etc. for matrices. Subscripted vectors such as $\mathbf{x_i}$ represent the $i^{\text{th}}$ block of $\mathbf{x}$. The $j^{\text{th}}$ component of the vectors $\mathbf{x}$ and $\mathbf{x_i}$ are indicated by $x_j$ and $x_{ij}$. We use $\mathbf{0}$ and $\mathbf{1}$ for the zero vector and vector of all ones, respectively, and $0$ and $I$ for the zero and identity matrices; in all cases the dimensions of these vectors and matrices can be discerned from the context.

Often we need to concatenate vectors and matrices. These concatenations may be column-wise or row-wise. We follow the convention of some high level programming languages, such as MATLAB, and use "," for adjoining vectors and matrices in a row and ";" for adjoining them in a column. Thus for instance for vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ the following are synonymous:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = (\mathbf{x}^\top, \mathbf{y}^\top, \mathbf{z}^\top)^\top = (\mathbf{x}; \mathbf{y}; \mathbf{z}).$$

If $\mathcal{A} \subseteq \Re^k$ and $\mathcal{B} \subseteq \Re^l$ then

$\mathcal{A} \times \mathcal{B} \stackrel{\text{def}}{=} \{(\mathbf{x}; \mathbf{y}) : \mathbf{x} \in \mathcal{A} \text{ and } \mathbf{y} \in \mathcal{B}\}$ is their Cartesian product.

For two matrices $A$ and $B$,

$$A \oplus B \stackrel{\text{def}}{=} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Let $\mathcal{K} \subseteq \Re^k$ be a closed, pointed (i.e. $\mathcal{K} \cap (-\mathcal{K}) = \{\mathbf{0}\}$) and convex cone with nonempty interior in $\Re^k$; in this article we exclusively work with such cones. It is well-known that $\mathcal{K}$ induces a partial order on $\Re^k$:

$$\mathbf{x} \succcurlyeq_{\mathcal{K}} \mathbf{y} \text{ if and only if } \mathbf{x} - \mathbf{y} \in \mathcal{K} \text{ and } \mathbf{x} \succ_{\mathcal{K}} \mathbf{y} \text{ if and only if } \mathbf{x} - \mathbf{y} \in \operatorname{int} \mathcal{K}$$

The relations $\preccurlyeq_{\mathcal{K}}$ and $\prec_{\mathcal{K}}$ are defined similarly. For each cone $\mathcal{K}$ the dual cone is defined by

$$\mathcal{K}^* = \left\{\mathbf{z} : \text{ for each } \mathbf{x} \in \mathcal{K}, \ \ \mathbf{x}^\top \mathbf{z} \geq 0\right\}.$$

Let $\mathcal{K}$ be the Cartesian product of several cones: $\mathcal{K} = \mathcal{K}_{n_1} \times \cdots \times \mathcal{K}_{n_r}$, where each $\mathcal{K}_{n_i} \subseteq \Re^{n_i}$. In such cases the vectors $\mathbf{x}$, $\mathbf{c}$ and $\mathbf{z}$ and the matrix $A$ are partitioned conformally, i.e.

$$\begin{array}{ll} \mathbf{x} = (\mathbf{x_1}; \ldots; \mathbf{x_r}) & \text{where } \mathbf{x_i} \in \Re^{n_i}, \\ \mathbf{z} = (\mathbf{z_1}; \ldots; \mathbf{z_r}) & \text{where } \mathbf{z_i} \in \Re^{n_i}, \\ \mathbf{c} = (\mathbf{c_1}; \ldots; \mathbf{c_r}) & \text{where } \mathbf{c_i} \in \Re^{n_i}, \\ A = (A_1, \ldots, A_r) & \text{where each } A_i \in \Re^{m \times n_i}. \end{array} \tag{1}$$

Throughout we take $r$ to be the number of blocks, $n = \sum_{i=1}^r n_i$ the dimension of the problem, and $m$ the number of rows in each $A_i$.

For each single block vector $\mathbf{x} \in \Re^n$ indexed from 0, we write $\bar{\mathbf{x}}$ for the sub-vector consisting of entries 1 through $n-1$; thus $\mathbf{x} = (x_0; \bar{\mathbf{x}})$. Also $\hat{\mathbf{x}} = (0; \bar{\mathbf{x}})$. Similarly for a matrix $A \in \Re^{m \times n}$ whose columns are indexed from 0, $\bar{A}$ refers to the sub-matrix of $A$ consisting of columns 1 through $n-1$.

Here we are primarily interested in the case where each cone $\mathcal{K}_i$ is the second-order cone:

$$\mathcal{Q}_n = \{\mathbf{x} = (x_0; \bar{\mathbf{x}}) \in \Re^n : x_0 \geq \|\bar{\mathbf{x}}\|\},$$

where $\|\cdot\|$ refers to the standard Euclidean norm, and $n$ is the dimension of $\mathcal{Q}_n$. If $n$ is evident from the context we drop it from the subscript. We refer to inequalities $\mathbf{x} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$ as second-order cone inequalities.

**Lemma 1** *$\mathcal{Q}$ is self-dual.*

Because of the special role played by the $0^{\text{th}}$ coordinate in second-order cones it is useful to define the reflection matrix

$$R_k \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix} \in \Re^{k\times k}$$

and the vector

$$\mathbf{e}_k \stackrel{\text{def}}{=} (1;\mathbf{0}) \in \Re^k.$$

We may often drop the subscripts if the dimension is evident from the context or if it is not relevant to the discussion.

**Definition 2** *The standard form Second-Order Cone Programming (SOCP) problem and its dual are*

| **Primal** | | **Dual** | |
|---|---|---|---|
| min | $\mathbf{c_1}^\top\mathbf{x_1}+\cdots+\mathbf{c_r}^\top\mathbf{x_r}$ | max | $\mathbf{b}^\top\mathbf{y}$ |
| s. t. | $A_1\mathbf{x_1}+\cdots+A_r\mathbf{x_r}=\mathbf{b}$ | s. t. | $A_i^\top\mathbf{y}+\mathbf{z_i}=\mathbf{c_i}$, *for* $i=1,\ldots,r$ |
| | $\mathbf{x_i}\succcurlyeq_{\mathcal{Q}}\mathbf{0}$, *for* $i=1,\ldots,r$ | | $\mathbf{z_i}\succcurlyeq_{\mathcal{Q}}\mathbf{0}$, $i=1,\ldots,r$ |

(2)

We make the following assumptions about the primal-dual pair (2):

**Assumption 1.** The $m$ rows of the matrix $A=(A_1,\ldots,A_r)$ are linearly independent.

**Assumption 2.** Both primal and dual problems are strictly feasible; i.e., there exists a primal-feasible vector $\mathbf{x}=(\mathbf{x_1};\ldots;\mathbf{x_r})$ such that $\mathbf{x_i}\succ_{\mathcal{Q}}\mathbf{0}$ for $i=1,\ldots,r$, and there exist a dual-feasible $\mathbf{y}$ and $\mathbf{z}=(\mathbf{z_1};\ldots;\mathbf{z_r})$ such that $\mathbf{z_i}\succ_{\mathcal{Q}}\mathbf{0}$, for $i=1,\ldots,r$.

We will see in §5 the complications that can arise if the second assumption does not hold.

Associated with each vector $\mathbf{x}\in\Re^n$ there is an *arrow-shaped* matrix $\operatorname{Arw}(\mathbf{x})$ defined as:

$$\operatorname{Arw}(\mathbf{x}) \stackrel{\text{def}}{=} \begin{pmatrix} x_0 & \bar{\mathbf{x}}^\top \\ \bar{\mathbf{x}} & x_0 I\end{pmatrix}$$

Observe that $\mathbf{x}\succcurlyeq_{\mathcal{Q}}\mathbf{0}$, $(\mathbf{x}\succ_{\mathcal{Q}}\mathbf{0})$ if and only if $\operatorname{Arw}(\mathbf{x})$ is positive semidefinite (positive definite), i.e., $\operatorname{Arw}(\mathbf{x})\succcurlyeq 0$ ($\operatorname{Arw}(\mathbf{x})\succ 0$). This is so because $\operatorname{Arw}(\mathbf{x})\succcurlyeq 0$ if and only if either $\mathbf{x}=\mathbf{0}$, or $x_0>0$ and the Schur complement $x_0-\bar{\mathbf{x}}^\top(x_0I)^{-1}\bar{\mathbf{x}}\geq 0$. Thus, SOCP is a special case of semidefinite programming. However, in this paper we argue that SOCP warrants its own study, and in particular requires special purpose algorithms.

For the cone $\mathcal{Q}$, let

$$\operatorname{bd}\mathcal{Q} \stackrel{\text{def}}{=} \{\mathbf{x}\in\mathcal{Q} : x_0=\|\bar{\mathbf{x}}\| \text{ and } \mathbf{x}\neq\mathbf{0}\}$$

denote the boundary of $\mathcal{Q}$ without the origin $\mathbf{0}$. Also, let

$$\text{int}\, \mathcal{Q} \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathcal{Q} : x_0 > \|\bar{\mathbf{x}}\|\}$$

denote the interior of $\mathcal{Q}$.

We also use $\mathcal{Q}$, Arw (.), $R$, and $\mathbf{e}$ in the block sense; that is if $\mathbf{x} = (\mathbf{x_1}; \cdots; \mathbf{x_r})$ such that $\mathbf{x_i} \in \Re^{n_i}$ for $i = 1, \ldots, r$, then

$$\begin{aligned}
&\mathcal{Q} \stackrel{\text{def}}{=} \mathcal{Q}_{n_1} \times \cdots \times \mathcal{Q}_{n_r} \\
&\text{Arw}\,(\mathbf{x}) \stackrel{\text{def}}{=} \text{Arw}\,(\mathbf{x_1}) \oplus \cdots \oplus \text{Arw}\,(\mathbf{x_n}) \\
&R \stackrel{\text{def}}{=} R_{n_1} \oplus \cdots \oplus R_{n_r} \\
&\mathbf{e} \stackrel{\text{def}}{=} (\mathbf{e_{n_1}}; \cdots; \mathbf{e_{n_r}}).
\end{aligned}$$

# 2 Formulating Problems as SOCPs

The standard form SOCP clearly looks very similar to the standard form linear program (LP):

$$\begin{array}{ll}
\min & \sum_{i=1}^{k} c_i x_i \\
\text{s.t.} & \sum_{i=1}^{k} x_i \mathbf{a_i} = \mathbf{b} \\
& x_i \geq 0, \quad \text{for } i = 1, \ldots, k,
\end{array}$$

where here the problem variables $x_i \in \Re, i = 1, \ldots, n$ and the objective function coefficients $c_i \in \Re, i = 1, \ldots, n$ are scalars and the constraint data $\mathbf{a_i} \in \Re^m, \quad i = 1, \ldots, n$ and $\mathbf{b} \in \Re^m$ are vectors. The non negativity constraints $x_i \geq 0, \quad i = 1, \ldots, k$, are just second-order cone constraints in spaces of dimension one. Hence, LP is a special case of SOCP. Also, since the second-order cone in $\Re^2$, $K = \{(x_0; x_1) \in \Re^2 \quad | \quad x_0 \geq |x_1|\}$, is a rotation of the nonnegative quadrant, it is clear that an SOCP in which all second-order cones are either one- or two-dimensional can be transformed into an LP.

Since second-order cones are convex sets, an SOCP is a convex programming problem. Also, if the dimension of a second-order cone is greater than two, it is not polyhedral, and hence in general, the feasible region of an SOCP is not polyhedral.

## 2.1 QPs and QCQPs

Like LPs, strictly convex quadratic programs (QPs) have polyhedral feasible regions and can be solved as SOCPs. Specifically, consider the strictly convex QP:

$$\begin{array}{ll}
\min & q(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{x}^\top Q \mathbf{x} + \mathbf{a}^\top \mathbf{x} + \beta, \\
& A\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0},
\end{array}$$

where $Q$ is a symmetric positive definite matrix, i.e., $Q \succ 0$, $Q = Q^\top$. Note that the objective function can be written as $q(\mathbf{x}) = \|\bar{\mathbf{u}}\|^2 + \beta - \frac{1}{4}\mathbf{a}^\top Q^{-1}\mathbf{a}$, where $\bar{\mathbf{u}} = Q^{1/2}\mathbf{x} + \frac{1}{2}Q^{-1/2}\mathbf{a}$. Hence, the above problem can be transformed into the SOCP:

$$\begin{array}{ll} \min & u_0 \\ \text{s.t.} & Q^{1/2}\mathbf{x} - \bar{\mathbf{u}} = \frac{1}{2}Q^{-1/2}\mathbf{a} \\ & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \quad (u_0; \bar{\mathbf{u}}) \succeq_Q \mathbf{0}. \end{array}$$

While both problems will have the same optimal solution, the square of the objective $u_0$ differs by $\beta - \frac{1}{4}\mathbf{a}^\top Q^{-1}\mathbf{a}$ from the original objective.

More generally, convex quadratically constrained quadratic programs (QCQPs) can be solved as SOCPs. To do so, we first observe that a QCQP can be expressed as the minimization of a linear function subject to convex quadratic constraints; i.e., as

$$\begin{array}{ll} \min & \mathbf{c}^\top \mathbf{x} \\ \text{s. t.} & q_i(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{x}^\top B_i^\top B_i \mathbf{x} + \mathbf{a_i}^\top \mathbf{x} + \beta_i \leq 0, \quad \text{for } i = 1, \ldots m, \end{array}$$

where $B_i \in \Re^{k_i \times n}$ and has rank $k_i, i = 1, \ldots, m$. To complete the reformulation, we observe that the convex quadratic constraint

$$q(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{x}^\top B^\top B\mathbf{x} + \mathbf{a}^\top \mathbf{x} + \beta \leq 0 \tag{3}$$

is equivalent to the second-order cone constraint $(u_0; \bar{\mathbf{u}}) \succeq_Q \mathbf{0}$, where

$$\bar{\mathbf{u}} = \begin{pmatrix} B\mathbf{x} \\ \frac{\mathbf{a}^\top \mathbf{x} + \beta + 1}{2} \end{pmatrix} \quad \text{and} \quad u_0 = \frac{1 - \mathbf{a}^\top \mathbf{x} - \beta}{2}.$$

## 2.2 Norm minimization problems

SOCP includes other classes of convex optimization problems as well. In particular, let $\overline{\mathbf{v}}_{\mathbf{i}} = A_i\mathbf{x} + \mathbf{b_i} \in \Re^{n_i}, \ i = 1, \ldots, r$. Then the following norm minimization problems can all be formulated and solved as SOCPs.

**a) Minimize the sum of norms:**

The problem min $\sum_{i=1}^m \|\overline{\mathbf{v}}_{\mathbf{i}}\|$ can be formulated as

$$\begin{array}{ll} \min & \sum_{i=1}^r v_{i0} \\ \text{s. t.} & A_i\mathbf{x} + \mathbf{b_i} = \overline{\mathbf{v}}_{\mathbf{i}} \quad \text{for } i = 1, \ldots, r \\ & \mathbf{v_i} \succeq_Q \mathbf{0}, \quad \text{for } i = 1, \ldots, r \end{array}$$

Observe that if we have nonnegative weights in the sum, the problem is still an SOCP.

The classical *Fermat-Weber* problem is a special case of the sum of norms problem. The Fermat-Weber problem considers where to place a facility so that the sum of the distances from this facility to a set of fixed locations is minimized. This problem is formulated as $\min_{\mathbf{x}} \sum_{i=1}^{k} \|\mathbf{d_i} - \mathbf{x}\|$, where the $\mathbf{d_i}$, $i = 1, \ldots, k$, are the fixed locations and $\mathbf{x}$ is the unknown facility location.

**b) Minimize the maximum of norms:**

The problem $\min\ \max_{1\le i\le r} \|\overline{\mathbf{v}}_{\mathbf{i}}\|$ has the SOCP formulation

$$\begin{array}{ll} \min & t \\ \text{s.t.} & A_i\mathbf{x} + \mathbf{b_i} = \overline{\mathbf{v}}_{\mathbf{i}} \quad \text{for } i = 1, \ldots, r \\ & (t; \overline{\mathbf{v}}_{\mathbf{i}}) \succcurlyeq_{\mathcal{Q}} \mathbf{0} \quad \text{for } i = 1, \ldots, r. \end{array}$$

**c) Minimize the sum of the $k$ largest norms:**

The problem $\min\ \sum_{i=1}^{k} \|\bar{\mathbf{v}}_{[i]}\|$, where $\|\bar{\mathbf{v}}_{[1]}\|$, $\|\bar{\mathbf{v}}_{[2]}\|, \ldots, \|\bar{\mathbf{v}}_{[r]}\|$ are the norms $\|\bar{\mathbf{v}}_1\|, \ldots, \|\bar{\mathbf{v}}_r\|$ sorted in nonincreasing order, has the SOCP formulation ([LVBL98])

$$\begin{array}{ll} \min & \sum_{i=1}^{r} u_i + kt \\ \text{s.t.} & A_i\mathbf{x} + \mathbf{b_i} = \overline{\mathbf{v}}_{\mathbf{i}} \quad \text{for } i = 1, \ldots, r \\ & \|\overline{\mathbf{v}}_{\mathbf{i}}\| \le u_i + t, \quad i = 1, \ldots, r \\ & u_i \ge 0, \quad i = 1, \ldots, r. \end{array}$$

## 2.3 Other SOCP-Representable problems and functions

If we rotate the second-order cone $\mathcal{Q}$ through an angle of forty-five degrees in the $x_0, x_1$-plane, we obtain the *rotated quadratic cone*

$$\hat{\mathcal{Q}} \stackrel{\text{def}}{=} \{\mathbf{x} = (x_0; x_1; \hat{\mathbf{x}}) \in \Re \times \Re \times \Re^{n_i - 2} \mid 2x_0x_1 \ge \|\hat{\mathbf{x}}\|^2, x_0 \ge 0, x_1 \ge 0\},$$

where $(x_1; \hat{\mathbf{x}}) \stackrel{\text{def}}{=} \bar{\mathbf{x}}$. This rotated quadratic cone is useful for transforming convex quadratic constraints into second-order cone constraints as we have already demonstrated. (See (3) and the construction immediately following it.) It is also very useful in converting problems with *restricted hyperbolic* constraints into SOCPs. Specifically, a constraint of the form: $\mathbf{w}^\top\mathbf{w} \le xy$, where $x \ge 0$, $y \ge 0$, $\mathbf{w} \in \Re^n$, $x, y \in \Re$, is equivalent to the second-order cone constraint

$$\left\| \begin{pmatrix} 2\mathbf{w} \\ x - y \end{pmatrix} \right\| \le x + y.$$

This observation is the basis for significantly expanding the set of problems that can be formulated as SOCPs beyond those that are norm minimization and QCQP problems. We now exhibit some examples of its use. This material is mostly based on the work of Nesterov and Nemirovski [NN94, Nem99] (see also Lobo et al. [LVBL98]). In fact, the first two of these references present a powerful calculus that applies standard convexity-preserving mappings to second-order cones to transform them into more complicated sets. In [Nem99], it is also shown that some convex sets that are not representable by a system of second-order cone and linear inequalities can be approximately represented by such systems to a high degree of accuracy by greatly increasing the number of variables and inequalities. Furthermore, Ben-Tal and Nemirovski [BTN01] show that a second-order cone inequality in $\Re^m$ can be approximated by linear inequalities to within $\epsilon$ accuracy using $\mathcal{O}\left(m \ln \frac{1}{\epsilon}\right)$ extra variables and inequalities.

**a) Minimize the harmonic mean of positive affine functions:**

The problem min $\sum_{i=1}^r 1/(\mathbf{a_i}^\top \mathbf{x} + \beta_i)$, where $\mathbf{a_i}^\top \mathbf{x} + \beta_i > 0$, $i = 1, \ldots, r$, can be formulated as:

$$\begin{array}{ll} \min & \sum_{i=1}^r u_i \\ \text{s.t.} & v_i = \mathbf{a_i}^\top \mathbf{x} + \beta_i, \quad i = 1, \ldots, r \\ & 1 \leq u_i v_i, \quad i = 1, \ldots, r \\ & u_i \geq 0, \quad i = 1, \ldots, r. \end{array}$$

**b) Logarithmic Tchebychev approximation:**

Consider the problem,

$$\min \max_{1 \leq i \leq r} \left| \ln(\mathbf{a_i}^\top \mathbf{x}) - \ln(b_i) \right|, \tag{4}$$

where $b_i > 0$, $i = 1, \ldots, r$, and $\ln(\mathbf{a_i}^\top \mathbf{x})$ is interpreted as $-\infty$ when $\mathbf{a_i}^\top \mathbf{x} \leq 0$. It can be formulated as an SOCP using the observation that

$$\left| \ln(\mathbf{a_i}^T \mathbf{x}) - \ln(b_i) \right| = \ln \max\left( \frac{\mathbf{a_i}^\top \mathbf{x}}{b_i}, \frac{b_i}{\mathbf{a_i}^\top \mathbf{x}} \right),$$

assuming that $\mathbf{a_i}^\top \mathbf{x} > 0$. Hence, (4) is equivalent to

$$\begin{array}{ll} \min & t \\ \text{s.t.} & 1 \leq \left(\mathbf{a_i}^\top \mathbf{x}/b_i\right) t \quad \text{for } i = 1, .., r \\ & \mathbf{a_i}^\top \mathbf{x}/b_i \leq t \quad \text{for } i = 1, \ldots, r, \\ & t \geq 0. \end{array}$$

**c) Inequalities involving the sum of quadratic/linear fractions:**

The inequality $\sum_{i=1}^r \frac{\|A_i \mathbf{x} + \mathbf{b_i}\|^2}{\mathbf{a_i}^\top \mathbf{x} + \beta_i} \leq t$, where for all $i$, $A_i \mathbf{x} + \mathbf{b_i} = \mathbf{0}$ if $\mathbf{a_i}^\top \mathbf{x} + \beta_i = 0$ and $0^2/0 = 0$, can be represented by the system

$$\begin{array}{l} \sum_{i=1}^r u_i \leq t, \\ \mathbf{w_i}^\top \mathbf{w_i} \leq u_i v_i, \quad i = 1, \ldots, r \\ \mathbf{w_i} = A_i \mathbf{x} + \mathbf{b_i}, \quad i = 1, \ldots, r \\ v_i = \mathbf{a_i}^\top \mathbf{x} + \beta_i \geq 0, \quad i = 1, \ldots, r. \end{array}$$

**d) Fractional quadratic functions:**

Problems involving fractional quadratic functions of the form $\mathbf{y}^\top A(\mathbf{s})^{-1}\mathbf{y}$, where $A(\mathbf{s}) = \sum_{i=1}^k s_i A_i$, $A_i \in \Re^{n\times n}, i = 1,\dots k$ are symmetric positive semidefinite matrices with a positive definite sum, $\mathbf{y} \in \Re^n$ and $\mathbf{s} \in \Re^k_+$, where $\Re^k_+$ ($\Re^k_{++}$) denotes the set of nonnegative (positive) vectors in $\Re^k$ can often be formulated as SOCPs. Specifically, let us consider the inequality constraint

$$\mathbf{y}^\top A(\mathbf{s})^{-1}\mathbf{y} \le t, \tag{5}$$

where $t \in \Re_+$. Under the assumption that $\mathbf{s} > \mathbf{0}$, which ensures that $A(\mathbf{s})$ is nonsingular (see [NN94] for the general case of $\mathbf{s} \ge \mathbf{0}$), we now show that $(\mathbf{y};\mathbf{s};t) \in \Re^n \times \Re^k_{++} \times \Re_+$ satisfies (5) if and only if there exist $\mathbf{w_i} \in \Re^{r_i}$ and $t_i \in \Re_+, i = 1,\dots,k$ such that

$$\begin{array}{l} \sum_{i=1}^k D_i^\top \mathbf{w_i} = \mathbf{y}, \\ \sum_{i=1}^k t_i \le t, \\ \mathbf{w_i}^\top \mathbf{w_i} \le s_i t_i, \quad i = 1,\dots,k, \end{array} \tag{6}$$

where for $i = 1,\dots,k$, $r_i = \operatorname{rank}(A_i)$ and $D_i$ is a $r_i \times n$ matrix such that $D_i^\top D_i = A_i$.

To prove this, first assume that $\mathbf{y}, \mathbf{s}$ and $t$ satisfy (5). Then defining $\mathbf{u}$ by $A(\mathbf{s})\mathbf{u} = \mathbf{y}$ and $\mathbf{w_i}$ by $\mathbf{w_i} = s_i D_i \mathbf{u}$, for $i = 1,\dots,k$, it follows that

$$\sum_{i=1}^k \frac{\mathbf{w_i}^\top \mathbf{w_i}}{s_i} = \mathbf{u}^\top A(\mathbf{s})\mathbf{u} = \mathbf{y}^\top A(\mathbf{s})^{-1}\mathbf{y} \le t$$

and $\sum_{i=1}^k D_i^\top \mathbf{w_i} = \sum_{i=1}^k \mathbf{s_i} D_i^\top D_i \mathbf{u} = A(\mathbf{s})\mathbf{u} = \mathbf{y}$. Therefore, letting $t_i = \frac{\mathbf{w_i}^\top \mathbf{w_i}}{s_i}, i = 1,\dots,k$ yields a solution to (6).

Now suppose that $(\mathbf{y};\mathbf{s};t) \in \Re^n \times \Re^k_{++} \times \Re_+$ and $\mathbf{w_i} \in \Re^{r_i}$ and $t_i \in \Re_+$, $i = 1,\dots,k$ is a solution to (6), or equivalently, that $(\mathbf{y};\mathbf{s};t) \in \Re^n \times \Re^k_{++} \times \Re_+$ and $\mathbf{w_i} \in \Re^{r_i}$, $i = 1,\dots,k$ is a solution to

$$\begin{array}{l} \sum_{i=1}^k D_i^\top \mathbf{w_i} = \mathbf{y}, \\ \sum_{i=1}^k \frac{\mathbf{w_i}^\top \mathbf{w_i}}{s_i} \le t. \end{array} \tag{7}$$

Consider the problem

$$\begin{array}{ll} \min & \sum_i \frac{\mathbf{w_i}^\top \mathbf{w_i}}{s_i} \\ \text{s.t.} & \sum_{i=1}^k D_i^\top \mathbf{w_i} = \mathbf{y}, \end{array} \tag{8}$$

for fixed $\mathbf{s} > \mathbf{0}$ and $\mathbf{y}$ that satisfy (7) for some $\mathbf{w_i}, i = 1,\dots,k$. From the KKT conditions for an optimal solution $\mathbf{w_i^*}$, $i = 1,\dots,k$ of this problem, there exists a vector $\mathbf{u} \in \Re^n$ such that $\mathbf{w_i^*} = s_i D_i \mathbf{u}$, $i = 1,\dots,k$, and, hence, the constraint in (8) implies that $A(\mathbf{s})\mathbf{u} = \mathbf{y}$. But (7) implies that the optimal value of the objective function in (8) $\sum_i \frac{\mathbf{w_i^*}^\top \mathbf{w_i^*}}{s_i} = \sum_i s_i \mathbf{u}^\top D_i^\top D_i \mathbf{u} = \mathbf{u}^\top A(\mathbf{s})\mathbf{u} \le t$, which shows that $(\mathbf{y};\mathbf{s};t)$ is a solution to (5).

**e) Maximize the geometric mean of nonnegative affine functions:**

To illustrate how to transform the problem max $\prod_{i=1}^{r}(\mathbf{a_i}^\top \mathbf{x} + \beta_i)^{1/r}$ into one with hyperbolic constraints, we show below how this is done for the case of $r = 4$.

$$\begin{array}{ll} \max & w_3 \\ \text{s.t.} & v_i = \mathbf{a_i}^\top \mathbf{x} + \beta_i \geq 0, \quad i = 1\ldots, 4 \\ & w_1^2 \leq v_1 v_2, \quad w_2^2 \leq v_3 v_4 \quad w_3^2 \leq w_1 w_2, \\ & w_1 \geq 0, \quad w_2 \geq 0. \end{array}$$

In formulating the geometric mean problem above as an SOCP, we have used the fact that an inequality of the form

$$t^{2^k} \leq s_1 s_2 \ldots s_{2^k} \tag{9}$$

for $t \in \Re$, and $s_1 \geq 0, \ldots, s_{2^k} \geq 0$ can be expressed by $2^{k-1}$ inequalities of the form $w_i^2 \leq u_i v_i$, where all new variables that are introduced are required to be nonnegative. If some of the variables are identical, fewer inequalities may be needed. Also, some variables $s_i$ may be constants. Other applications of (9) are:

**f) Inequalities involving rational powers :**

To illustrate that systems of inequalities of the form

$$\prod_{i=1}^{r} x_i^{-\pi_i} \leq t, \quad x_i \geq 0, \pi_i > 0, \qquad \text{for } i = 1, \ldots, r, \tag{10}$$

$$-\prod_{i=1}^{r} x_i^{\pi_i} \leq t, \quad x_i \geq 0, \pi_i > 0, \qquad \text{for } i = 1, \ldots, r, \quad \sum_{i=1}^{r} \pi_i \leq 1 \tag{11}$$

where $\pi_i$ are rational numbers, have second-order cone representations, consider the two examples:

$$x_1^{-5/6} x_2^{-1/3} x_3^{-1/2} \leq t, \quad x_i \geq 0, \quad \text{for } i = 1, 2, 3, \tag{12}$$

$$-x_1^{1/5} x_2^{2/5} x_3^{1/5} \leq t, \quad x_i \geq 0, \quad \text{for } i = 1, 2, 3. \tag{13}$$

The first inequality in (12) is equivalent to the multinomial inequality

$$1^{16} = 1 \leq x_1^5 x_2^2 x_3^3 t^6,$$

which in turn is equivalent to the system

$$\begin{array}{lll} w_1^2 \leq x_1 x_3, & w_2^2 \leq x_3 w_1, & w_3^2 \leq x_2 t, \\ w_4^2 \leq w_2 w_3, & w_5^2 \leq x_1 t, & 1^2 \leq w_4 w_5. \end{array}$$

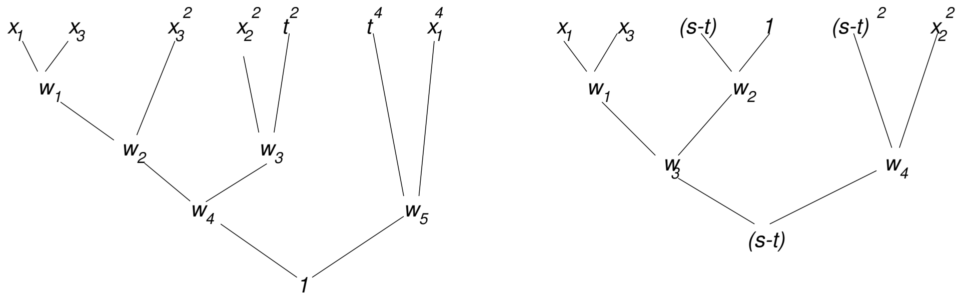
By introducing the variable $s \geq 0$, where $s - t \geq 0$, the first inequality in (13) is equivalent to

$$s - t \leq x_1^{1/5} x_2^{2/5} x_3^{1/5}, \quad s \geq 0, \quad s - t \geq 0,$$

which in turn is equivalent to $(s-t)^8 \le x_1 x_2^2 x_3 (s-t)^3 1$, $s \ge 0$, $s-t \ge 0$. Finally, this can be represented by the system:

$$\begin{array}{llll} w_1^2 \le x_1 x_3, & w_2^2 \le (s-t)1, & w_3^2 \le w_1 w_2, & s \ge 0, \\ w_4^2 \le (s-t)x_2, & (s-t)^2 \le w_3 w_4, & s-t \ge 0. & \end{array}$$

Both of these formulations are illustrated below:




From these figures it is apparent that the second-order cone representations are not unique.

**g) Inequalities involving $p$-norms:**

Inequalities

$$|x|^p \le t \qquad x \in \Re, \tag{14}$$

$$\|\mathbf{x}\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} \le t, \tag{15}$$

involving the $p^{\text{th}}$ power of the absolute value of a scalar variable $\mathbf{x} \in \Re$ and those involving the $p$-norm of a vector $\mathbf{x} \in \Re^n$, where $p = l/m$ is a positive rational number with $l \ge m$, have second-order cone representations. To see this, note that inequality (14) is equivalent to $|x| \le t^{m/l}$ and $t \ge 0$ and, hence, to

$$-t^{m/l} \le -x, \qquad -t^{m/l} \le x, \qquad t \ge 0. \tag{16}$$

But the first two inequalities in (16) are of the form (11), where the product consists of only a single term. Thus, they can be represented by a set of second-order cone inequalities.

The $p$-norm inequality,

$$\Big(\sum_{i=1}^n |x_i|^{l/m}\Big)^{m/l} \le t,$$

is equivalent to

$$\begin{aligned}
&-t^{\frac{l-m}{l}} s_i^{\frac{m}{l}} \le -x_i, \qquad -t^{\frac{l-m}{l}} s_i^{m/l} \le x_i, \qquad s_i \ge 0, \quad \text{for } i = 1, \ldots, n, \\
&\sum_{i=1}^{n} s_i \le t, \qquad t \ge 0,
\end{aligned}$$

and hence to a set of second-order cone and linear inequalities.

**h) Problems involving pairs of quadratic forms:**

We consider here, as an illustration of such problems, the problem of finding the smallest ball $S = \{\mathbf{x} \in \Re^n \mid \|\mathbf{x}-\mathbf{a}\| \le \rho\}$, that contains the ellipsoids $\mathcal{E}_1, \ldots, \mathcal{E}_k$, where

$$\mathcal{E}_i \stackrel{\text{def}}{=} \{\mathbf{x} \in \Re^n \mid \mathbf{x}^\top A_i \mathbf{x} + 2\mathbf{b_i}^\top \mathbf{x} + c_i \le 0\}, \quad \text{for } i = 1, ..., k.$$

Applying the so-called $S$ procedure (see for example, [BGFB94]) $S$ contains the $\mathcal{E}_1, \ldots, \mathcal{E}_k$ if and only if there are nonnegative numbers $\tau_1, \ldots, \tau_k$ such that

$$M_i \stackrel{\text{def}}{=} \begin{pmatrix} \tau_i A_i - I & \tau_i \mathbf{b_i} + \mathbf{a} \\ \tau_i \mathbf{b_i}^\top + \mathbf{a}^\top & \tau_i c_i + \rho^2 - \mathbf{a}^\top \mathbf{a} \end{pmatrix} \succcurlyeq 0, \qquad \text{for } i = 1, \ldots, k.$$

Let $A_i = Q_i \Lambda_i Q_i^\top$ be the spectral decomposition of $A_i$, $\Lambda_i = \mathrm{Diag}(\lambda_{i1}; \ldots; \lambda_{in})$, $t = \rho^2$ and $\mathbf{v_i} = Q_i^\top(\tau_i \mathbf{b_i} + \mathbf{a})$, for $i = 1, \ldots, k$. Then

$$\bar{M}_i \stackrel{\text{def}}{=} \begin{pmatrix} Q_i^\top & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix} M_i \begin{pmatrix} Q_i & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix} = \begin{pmatrix} \tau_i \Lambda_i - I & \mathbf{v_i} \\ \mathbf{v_i}^\top & \tau_i c_i + t - \mathbf{a}^\top \mathbf{a} \end{pmatrix},$$

for $i = 1, \ldots, k$, and $M_i \succcurlyeq 0$ if and only if $\bar{M}_i \succcurlyeq 0$. But the latter holds if and only if $\tau_i \ge \frac{1}{\lambda_{\min}(A_i)}$, i.e., $\tau_i \lambda_{ij} - 1 \ge 0$ for all $i, j$, $v_{ij} = 0$ if $\tau_i \lambda_{ij} - 1 = 0$ and the Schur complement of the columns and rows of $\bar{M}_i$ that are not zero,

$$\tau_i c_i + t - \mathbf{a}^\top \mathbf{a} - \sum_{\tau_i \lambda_{ij} > 1} \frac{v_{ij}^2}{(\tau_i \lambda_{ij} - 1)} \ge 0. \tag{17}$$

If we define $\mathbf{s_i} = (s_{i1}; \ldots; s_{in})$, where $s_{ij} = \frac{v_{ij}^2}{\tau_i \lambda_{ij} - 1}$, for all $j$ such that $\tau_i \lambda_{ij} > 1$ and $s_{ij} = 0$, otherwise, then (17) is equivalent to

$$t \ge \mathbf{a}^\top \mathbf{a} - \tau_i c_i + \mathbf{1}^\top \mathbf{s_i}.$$

Since we are minimizing $t$, we can relax the definition of $s_{ij}$, replacing it by $v_{ij}^2 \le s_{ij}(\tau_i \lambda_{ij} - 1)$ and $s_{ij} \ge 0$, for $j = 1, \ldots, n$, and $i = 1, \ldots, k$. Combining all of the above yields the following formulation involving only linear and restricted hyperbolic constraints:

$$\begin{array}{ll}
\min & t \\
\text{s.t.} & \mathbf{v_i} = Q_i^\top(\tau_i \mathbf{b_i} + \mathbf{a}), \quad \text{for } i = 1, \ldots, k, \\
& v_{ij}^2 \le s_{ij}(\tau_i \lambda_{ij} - 1), \quad \text{for } i = 1, \ldots, k \quad \text{and } j = 1, \ldots, n, \\
& \mathbf{a}^\top \mathbf{a} \le t + \tau_i c_i - \mathbf{1}^\top \mathbf{s_i}, \quad \text{for } i = 1, \ldots, k, \\
& \tau_i \ge \frac{1}{\lambda_{\min}(A_i)}, \ \mathbf{s_i} \ge \mathbf{0}, \quad \text{for } i = 1, \ldots, k.
\end{array}$$

This problem can be transformed into an SOCP with $kn$ 3-dimensional and $k\,(n+2)$-dimensional second-order cone inequalities. It illustrates the usefulness of the Schur complement in transforming some linear matrix inequalities into systems of linear and second-order cone inequalities. Another application of this technique can be found in [GI01]. Note that the Schur complement is frequently applied in the reverse direction to convert second-order cone inequalities into linear matrix inequalities, which, as we have stated earlier, is not advisable from a computational point of view. Finally, this example demonstrates that although it may be possible to formulate an optimization problem as an SOCP, this task may be far from trivial.

The above classes of optimization problems encompass many real world applications in addition to those that can be formulated as LPs and convex QPs. These include facility location and Steiner tree problems [ACO94, XY97], grasping force optimization [BHM96, BFM97] and image restoration problems. Engineering design problems that can be solved as SOCPs include antenna array design [LB97], various finite impulse response filter design problems [LVBL98, WBV98], and truss design [BTN92]. In the area of financial engineering, portfolio optimization with loss risk constraints [LVBL98], robust multistage portfolio optimization [BTN99] and other robust portfolio optimization problems [GI01] lead to SOCPs. In the next section we give two examples of robust optimization problems that can be formulated as SOCPs.

# 3 Robust Least Squares and Robust Linear Programming

The determination of *robust* solutions to optimization problems has been an important area of activity in the field of control theory for some time. Recently, the idea of robustness has been introduced into the fields of mathematical programming and least squares. In this section, we show that the so-called robust counterparts of a least squares problem and a linear program can both be formulated as SOCPs.

## 3.1 Robust Least Squares

There is an extensive body of research on sensitivity analysis and regularization procedures for least squares problems. Some of the recent work in this area has focused on finding robust solutions to such problems when the problem data is uncertain but known to fall in some bounded region. Specifically, El Ghaoui and Lebret [GL97] consider the following *robust least squares* (RLS) problem. Given an over-determined set of equations

$$A\mathbf{x} \approx \mathbf{b}, \quad A \in \Re^{m\times n}, \tag{18}$$

where $[A, \mathbf{b}]$ is subject to unknown but bounded errors

$$\left\|[\triangle A, \triangle \mathbf{b}]\right\|_F \leq \rho, \tag{19}$$

and $\|B\|_F$ denotes the Frobenius norm of the matrix $B$, determine the vector $\mathbf{x}$ that minimizes the Euclidean norm of the worst-case residual vector corresponding to the system (18); i.e., solve the problem

$$\min_{\mathbf{x}} \max\Big\{ \|(A+\triangle A)\mathbf{x}-(\mathbf{b}+\triangle\mathbf{b})\| \;\Big|\; \|[\triangle A, \triangle\mathbf{b}]\|_F \le \rho \Big\}. \tag{20}$$

For a given vector $\mathbf{x}$, let us define

$$r(A,\mathbf{b},\mathbf{x}) \stackrel{\text{def}}{=} \max\Big\{ \|(A+\triangle A)\mathbf{x}-(\mathbf{b}+\triangle\mathbf{b})\| \;\Big|\; \|[\triangle A, \triangle\mathbf{b}]\|_F \le \rho \Big\}.$$

By the triangle inequality

$$\|(A+\triangle A)\mathbf{x}-(\mathbf{b}+\triangle\mathbf{b})\| \le \|A\mathbf{x}-\mathbf{b}\| + \left\|(\triangle A, -\triangle\mathbf{b})\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\|.$$

It then follows from properties of the Frobenius norm and the bound in (19) that

$$\left\|(\triangle A, -\triangle\mathbf{b})\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\| \le \|(\triangle A, \triangle\mathbf{b})\|_F \left\|\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\| \le \rho \left\|\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\|.$$

But for the choice $(\triangle A, -\triangle\mathbf{b}) = \mathbf{u}\mathbf{v}^\top$, where

$$\mathbf{u} = \begin{cases} \rho\frac{A\mathbf{x}-\mathbf{b}}{\|A\mathbf{x}-\mathbf{b}\|}, & \text{if } A\mathbf{x}-\mathbf{b}\neq 0 \\ \text{any vector } \in \Re^m \text{ of norm } \rho, & \text{otherwise} \end{cases} \qquad \text{and} \qquad v = \frac{\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}}{\left\|\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\|},$$

which satisfies (19), $A\mathbf{x}-b$ is a multiple of $(\Delta A, -\Delta\mathbf{b})\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}$, and

$$\left\|(\Delta A, -\Delta\mathbf{b})\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\| = \|\mathbf{u}\| \times \left\|\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\| = \rho\left\|\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\|.$$

Hence,

$$r(A,\mathbf{b},\mathbf{x}) = \|A\mathbf{x}-\mathbf{b}\| + \rho\left\|\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\|,$$

and robust least squares problem (20) is equivalent to a sum of norms minimization problem. Therefore, as we have observed earlier, this can be formulated as the SOCP

$$\begin{array}{ll} \min & \lambda + \rho\tau \\ \text{s.t.} & \|A\mathbf{x}-\mathbf{b}\| \le \lambda \\ & \left\|\begin{pmatrix}\mathbf{x}\\1\end{pmatrix}\right\| \le \tau. \end{array}$$

El Ghaoui and Lebret show that if the singular value decomposition of $A$ is available, then this SOCP can be reduced to the minimization of a smooth one-dimensional function. However, if additional constraints on $\mathbf{x}$, such as non-negativity, etc., are present, such a reduction is not possible. But as long as these additional constraints are either linear or quadratic, the problem can still be formulated as an SOCP.

### 3.2 Robust Linear Programming

The idea of finding a robust solution to an LP whose data is uncertain, but known to lie in some given set, has recently been proposed by Ben-Tal and Nemirovski [BTN99, BTN98]. Specifically, consider the LP

$$\begin{array}{ll} \min & \hat{\mathbf{c}}^\top \hat{\mathbf{x}} \\ \text{s.t.} & \hat{A}\hat{\mathbf{x}} \leq \mathbf{b}. \end{array}$$

where the constraint data $\hat{A} \in \Re^{m\times n}$ and $\mathbf{b} \in \Re^m$ are not known exactly. Note that we can handle uncertainty in the objective function coefficients by introducing an extra variable $z$, imposing the constraint $\hat{\mathbf{c}}^\top \hat{\mathbf{x}} \leq z$ and minimizing $z$. To simplify the presentation let us rewrite the above LP in the form

$$\begin{array}{ll} \min & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{0}, \\ & \boldsymbol{\delta}^\top \mathbf{x} = -1, \end{array} \tag{21}$$

where $A = [\hat{A}, \mathbf{b}]$, $\mathbf{c}^\top = (\hat{\mathbf{c}}^\top; 0)$, $\mathbf{x} = (\hat{\mathbf{x}}; \xi)$, and $\boldsymbol{\delta} = (\mathbf{0}; 1)$. Ben-Tal and Nemirovski treat the case where the uncertainty set for the data $A$ is the Cartesian product of ellipsoidal regions, one for each of the $m$ rows $\mathbf{a_i}^\top$ of $A$ centered at some given row vector $\hat{\mathbf{a}}_\mathbf{i}^\top \in \Re^{n+1}$. This means that, for all $i$, each vector $\mathbf{a_i}$ is required to lie in the ellipsoid $\mathcal{E}_i \stackrel{\text{def}}{=} \{\mathbf{a_i} \in \Re^{n+1} \mid \mathbf{a_i} = \hat{\mathbf{a}}_\mathbf{i} + B_i\mathbf{u}, \|\mathbf{u}\| \leq 1\}$, where $B_i$ is a $(n+1)\times(n+1)$ symmetric positive semidefinite matrix. For fixed $\mathbf{x}$, $\mathbf{a}_i^\top \mathbf{x} \leq 0$, $\forall \mathbf{a_i} \in \mathcal{E}_i$ if and only if $\max\{(\hat{\mathbf{a}}_\mathbf{i} + B_i\mathbf{u})^\top \mathbf{x} \mid \|\mathbf{u}\| \leq 1\} \leq 0$. But $\max\{(\hat{\mathbf{a}}_\mathbf{i}^\top \mathbf{x} + \mathbf{u}^\top B_i \mathbf{x}) \mid \|\mathbf{u}\| \leq 1\} = \hat{\mathbf{a}}_\mathbf{i}^\top \mathbf{x} + \|B_i\mathbf{x}\|$, for the choice $\mathbf{u} = \frac{B_i\mathbf{x}}{\|B_i\mathbf{x}\|}$, if $B_i\mathbf{x} \neq \mathbf{0}$, and for any $\mathbf{u}$ with $\|\mathbf{u}\| = 1$, if $B_i\mathbf{x} = \mathbf{0}$. Hence, for the above ellipsoidal uncertainty set, the robust counterpart of the LP (21) is

$$\begin{array}{lll} \min & \mathbf{c}^\top \mathbf{x} & \\ \text{s.t.} & \|B_i\mathbf{x}\| \leq -\hat{\mathbf{a}}_\mathbf{i}^\top \mathbf{x}, & i = 1, \ldots, m \\ & \boldsymbol{\delta}^\top \mathbf{x} = -1. & \end{array} \tag{22}$$

Ben-Tal and Nemirovski [BTN99] have also shown that the robust counterpart of an SOCP with ellipsoidal uncertainty sets can be formulated as a semidefinite program.

# 4 Algebraic Properties of Second-Order Cones

There is a particular algebra associated with second-order cones, the understanding of which sheds light on all aspects of the SOCP problem, from duality and complementarity properties, to conditions of non-degeneracy and ultimately to the design and analysis of interior point algorithms. This algebra is well-known and is a special case of a so-called *Euclidean Jordan algebra*, see the text of Faraut and Korány [FK94] for a comprehensive study. A Euclidean Jordan algebra is a framework within which algebraic properties of symmetric matrices are generalized. Both the Jordan algebra of the symmetric positive semidefinite cone (under the multiplication $X \circ Y = (XY + YX)/2$) and the algebra to be described below are special cases of a Euclidean Jordan algebra. This association unifies the analysis of semidefinite programming and SOCP. We do not treat a general Euclidean Jordan algebra, but rather focus entirely on the second-order cone and its associated algebra.

For now we assume that all vectors consist of a single block: $\mathbf{x} = (x_0; \bar{\mathbf{x}}) \in \Re^n$. For two vectors $\mathbf{x}$ and $\mathbf{y}$ define the following multiplication:

$$\mathbf{x} \circ \mathbf{y} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{x}^\top \mathbf{y} \\ x_0 y_1 + y_0 x_1 \\ \vdots \\ x_0 y_n + y_0 x_n \end{pmatrix} = \operatorname{Arw}(\mathbf{x})\, \mathbf{y} = \operatorname{Arw}(\mathbf{x}) \operatorname{Arw}(\mathbf{y})\, \mathbf{e}.$$

In this section we study the properties of the algebra $(\Re^n, \circ)$. Here are some of the prominent relations involving the binary operation $\circ$.

1. Since in the product $\mathbf{z} = \mathbf{x} \circ \mathbf{y}$ each component of $\mathbf{z}$ is a bilinear function of $\mathbf{x}$ and $\mathbf{y}$ the distributive law holds:
$$\mathbf{x} \circ (\alpha \mathbf{y} + \beta \mathbf{z}) = \alpha \mathbf{x} \circ \mathbf{y} + \beta \mathbf{x} \circ \mathbf{z} \quad \text{and} \quad (\alpha \mathbf{y} + \beta \mathbf{z}) \circ \mathbf{x} = \alpha \mathbf{y} \circ \mathbf{x} + \beta \mathbf{z} \circ \mathbf{x}$$
for all $\alpha, \beta \in \Re$.
2. $\mathbf{x} \circ \mathbf{y} = \mathbf{y} \circ \mathbf{x}$.
3. The vector $\mathbf{e} = (1; \mathbf{0})$ is the unique identity element: $\mathbf{x} \circ \mathbf{e} = \mathbf{x}$.
4. Let $\mathbf{x}^2 = \mathbf{x} \circ \mathbf{x}$. Then matrices $\operatorname{Arw}(\mathbf{x})$ and $\operatorname{Arw}(\mathbf{x}^2)$ commute. Therefore for all $\mathbf{y} \in \Re^n$
$$\mathbf{x} \circ (\mathbf{x}^2 \circ \mathbf{y}) = \mathbf{x}^2 \circ (\mathbf{x} \circ \mathbf{y}).$$
5. $\circ$ is not associative for $n > 2$. However, in the product of $k$ copies of a vector $\mathbf{x}$: $\mathbf{x} \circ \mathbf{x} \circ \cdots \circ \mathbf{x}$, the order in which multiplications are carried out does not matter. Thus $\circ$ is *power associative* and one can unambiguously write $\mathbf{x}^p$ for integral values of $p$. Furthermore, $\mathbf{x}^p \circ \mathbf{x}^q = \mathbf{x}^{p+q}$.

The first three assertions are obvious. The fourth one can be proved by inspection. We will prove the last assertion below after introducing the spectral properties of this algebra.

Next consider the *cone of squares* with respect to $\circ$:

$$\mathcal{L} = \{\mathbf{x}^2 : \mathbf{x} \in \Re^n\}.$$

We shall prove $\mathcal{L} = \mathcal{Q}$. First notice that if $\mathbf{x} = \mathbf{y}^2$ then

$$x_0 = \|\mathbf{y}\|^2 \quad \text{and} \quad \bar{\mathbf{x}} = 2 y_0 \bar{\mathbf{y}}.$$

Since clearly

$$x_0 = y_0^2 + \|\bar{\mathbf{y}}\|^2 \geq 2 y_0 \|\bar{\mathbf{y}}\| = \|\bar{\mathbf{x}}\|,$$

we conclude that $\mathcal{L} \subseteq \mathcal{Q}$. Now let $\mathbf{x} \in \mathcal{Q}$. We need to show that there is a $\mathbf{y} \in \Re^n$ such that $\mathbf{y}^2 = \mathbf{x}$; that is we need to show that the system of equations

$$x_0 = \|\mathbf{y}\|^2, \qquad x_i = 2y_0 y_i, \quad i = 1, \ldots, n-1,$$

has at least a real solution. First assume $y_0 \neq 0$. Then by substituting $y_i = x_i/(2y_0)$ in the $0^{\text{th}}$ equation we get the quartic equation

$$4y_0^4 - 4x_0 y_0^2 + \|\bar{\mathbf{x}}\|^2 = 0.$$

This equation has up to four solutions

$$y_0 = \pm\sqrt{\frac{x_0 \pm \sqrt{x_0^2 - \|\bar{\mathbf{x}}\|^2}}{2}},$$

all of them real because $x_0 \geq \|\bar{\mathbf{x}}\|$. Notice that one of them gives a $\mathbf{y}$ in $\mathcal{Q}$. In the algebra $(\Re^n, \circ)$, all elements $\mathbf{x} \in \operatorname{int} \mathcal{Q}$ have four square roots, except for multiples of the identity (where $\bar{\mathbf{x}} = \mathbf{0}$). In this case if $y_0 = 0$ then the $y_i$ can be arbitrarily chosen, as long as $\|\bar{\mathbf{y}}\| = 1$. In other words, the identity has infinitely many square roots (assuming $n > 2$). Two of them are $\pm\mathbf{e}$. All others are of the form $(0; \mathbf{q})$ with $\|\mathbf{q}\| = 1$. Elements on $\operatorname{bd} \mathcal{Q}$ have only two square roots, one of which is in $\operatorname{bd} \mathcal{Q}$, and those outside of $\mathcal{Q}$ have none. Thus every $\mathbf{x} \in \mathcal{Q}$ has a unique square root in $\mathcal{Q}$.

One can easily verify the following quadratic identity for $\mathbf{x}$:

$$\mathbf{x}^2 - 2x_0\mathbf{x} + (x_0^2 - \|\bar{\mathbf{x}}\|^2)\mathbf{e} = \mathbf{0}.$$

The polynomial

$$p(\lambda, \mathbf{x}) \stackrel{\text{def}}{=} \lambda^2 - 2x_0\lambda + (x_0^2 - \|\bar{\mathbf{x}}\|^2)$$

is the *characteristic polynomial* of $\mathbf{x}$ and the two roots of $p(\lambda, \mathbf{x})$, $x_0 \pm \|\bar{\mathbf{x}}\|$ are the *eigenvalues* of $\mathbf{x}$. Characteristic polynomials and eigenvalues in this algebra play much the same role as their counterparts in symmetric matrix algebra, except that the situation is simpler here. Every element has only two real eigenvalues and they can be easily calculated.

Let us also examine the following identity:

$$\mathbf{x} = \frac{1}{2}(x_0 + \|\bar{\mathbf{x}}\|)\begin{pmatrix} 1 \\ \frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|} \end{pmatrix} + \frac{1}{2}(x_0 - \|\bar{\mathbf{x}}\|)\begin{pmatrix} 1 \\ -\frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|} \end{pmatrix}. \tag{23}$$

Defining

$$\mathbf{c_1} \stackrel{\text{def}}{=} \frac{1}{2}\begin{pmatrix} 1 \\ \frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|} \end{pmatrix}, \quad \mathbf{c_2} \stackrel{\text{def}}{=} \frac{1}{2}\begin{pmatrix} 1 \\ -\frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|} \end{pmatrix}, \quad \lambda_1 = x_0 + \|\bar{\mathbf{x}}\|, \quad \lambda_2 = x_0 - \|\bar{\mathbf{x}}\|, \tag{24}$$

it follows that (23) can be written as

$$\mathbf{x} = \lambda_1\mathbf{c_1} + \lambda_2\mathbf{c_2} \tag{25}$$

Observe that

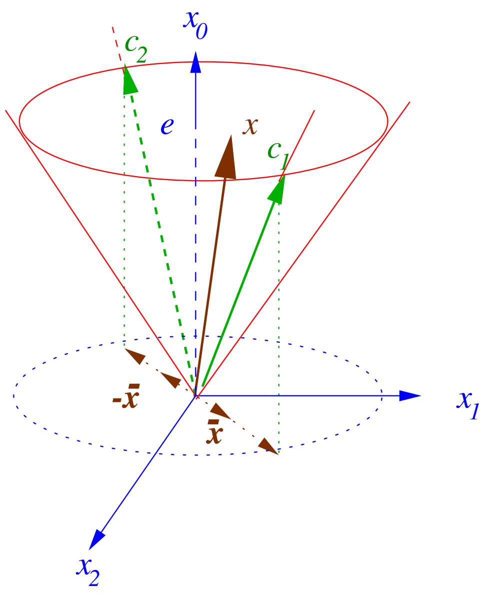
$$\mathbf{c_1} \circ \mathbf{c_2} = \mathbf{0}, \tag{26}$$
$$\mathbf{c_1}^2 = \mathbf{c_1} \quad \text{and} \quad \mathbf{c_2}^2 = \mathbf{c_2}, \tag{27}$$
$$\mathbf{c_1} + \mathbf{c_2} = \mathbf{e}, \tag{28}$$
$$\mathbf{c_1} = R\mathbf{c_2} \quad \text{and} \quad \mathbf{c_2} = R\mathbf{c_1}, \tag{29}$$
$$\mathbf{c_1}, \mathbf{c_2} \in \text{bd}\, \mathcal{Q}. \tag{30}$$

Equation (25) is the *spectral decomposition* of $\mathbf{x}$. It is analogous to its counterpart in symmetric matrix algebra:

$$X = Q\Lambda Q^\top = \lambda_1 \mathbf{q_1}\mathbf{q_1}^\top + \cdots + \lambda_n \mathbf{q_n}\mathbf{q_n}^\top,$$

where $\Lambda$ is a diagonal matrix with $\lambda_i$ the $i^{\text{th}}$ diagonal entry, and $Q$ is orthogonal with column $\mathbf{q_i}$ as the eigenvector corresponding to $\lambda_i$, for $i = 1, \ldots, n$. Any pair of vectors $\{\mathbf{c_1}, \mathbf{c_2}\}$ that satisfies properties (26),(27), and (28) is called a *Jordan frame*. It can be shown that every Jordan frame $\{\mathbf{c_1}, \mathbf{c_2}\}$ is of the form (24), that is Jordan frames are pairs of vectors $\left(\frac{1}{2}; \pm\bar{\mathbf{c}}\right)$ with $\|\bar{\mathbf{c}}\| = \frac{1}{2}$.

Jordan frames play the role of the sets of rank one matrices $\mathbf{q_i}\mathbf{q_i}^\top$ formed from orthogonal systems of eigenvectors in symmetric matrix algebra.




We can also define the *trace* and *determinant* of $\mathbf{x}$:

$$\text{tr}(\mathbf{x}) = \lambda_1 + \lambda_2 = 2x_0 \quad \text{and}$$
$$\det(\mathbf{x}) = \lambda_1\lambda_2 = x_0^2 - \|\bar{\mathbf{x}}\|^2.$$

We are now in a position to prove that $(\Re^n, \circ)$ is power associative. Let $\mathbf{x} = \lambda_1\mathbf{c_1} + \lambda_2\mathbf{c_2}$ be the spectral decomposition of $\mathbf{x}$. For nonnegative integer $p$ let us define

$$\mathbf{x}^{\circ 0} \stackrel{\text{def}}{=} \mathbf{e}, \quad \mathbf{x}^{\circ p} \stackrel{\text{def}}{=} \mathbf{x} \circ \mathbf{x}^{\circ(p-1)}, \quad \mathbf{x}^p \stackrel{\text{def}}{=} \lambda_1^p\mathbf{c_1} + \lambda_2^p\mathbf{c_2}$$

It suffices to show that $\mathbf{x}^{\circ p} = \mathbf{x}^p$. Since clearly the assertion is true for $p = 0$, we proceed by induction. Assuming that the assertion is true for $k$, we have

$$\begin{aligned}
\mathbf{x}^{\circ(k+1)} &= \mathbf{x} \circ \mathbf{x}^{\circ k} = (\lambda_1\mathbf{c_1} + \lambda_2\mathbf{c_2}) \circ (\lambda_1^k\mathbf{c_1} + \lambda_2^k\mathbf{c_2}) \\
&= \lambda_1^{k+1}\mathbf{c_1}^2 + \lambda_2^{k+1}\mathbf{c_2}^2 + \lambda_1\lambda_2^k\mathbf{c_1} \circ \mathbf{c_2} + \lambda_1^k\lambda_2\mathbf{c_2} \circ \mathbf{c_1} \\
&= \lambda^{k+1}\mathbf{c_1} + \lambda_2^{k+1}\mathbf{c_2},
\end{aligned}$$

since $\mathbf{c_1} \circ \mathbf{c_2} = \mathbf{0}$ and $\mathbf{c_i}^2 = \mathbf{c_i}$ for $i = 1, 2$. Similarly $\mathbf{x}^p \circ \mathbf{x}^q = \mathbf{x}^{p+q}$ follows from the spectral decompositions of $\mathbf{x}^p$ and $\mathbf{x}^q$ and $\mathbf{x}^{p+q}$. (We should remark that the proof given here is specific to $(\Re^n, \circ)$ and does not carry over to general Euclidean Jordan algebras.)

It is now obvious that $\mathbf{x} \in \mathcal{Q}$ if and only if $\lambda_{1,2}(\mathbf{x}) \geq 0$, and $\mathbf{x} \in \operatorname{int} \mathcal{Q}$ if and only if $\lambda_{1,2} > 0$. In analogy to matrices, we call $\mathbf{x} \in \mathcal{Q}$ *positive semidefinite* and $\mathbf{x} \in \operatorname{int} \mathcal{Q}$ *positive definite*.

The spectral decomposition opens up a convenient way of extending any real valued continuous function to the algebra of the second-order cone. Unitarily invariant norms may also be defined similarly to the way they are defined on symmetric matrices. For example, one can define analogs of Frobenius and 2 norms, inverse, square root or essentially any arbitrary power of $\mathbf{x}$ using the spectral decomposition of $\mathbf{x}$:

$$\|\mathbf{x}\|_F \stackrel{\text{def}}{=} \sqrt{\lambda_1^2 + \lambda_2^2} = \sqrt{2}\|\mathbf{x}\|, \tag{31}$$

$$\|\mathbf{x}\|_2 \stackrel{\text{def}}{=} \max\{|\lambda_1|, |\lambda_2|\} = |x_0| + \|\bar{\mathbf{x}}\|, \tag{32}$$

$$\mathbf{x}^{-1} \stackrel{\text{def}}{=} \lambda_1^{-1}\mathbf{c_1} + \lambda_2^{-1}\mathbf{c_2} = \frac{R\mathbf{x}}{\det(\mathbf{x})} \text{ if } \det(\mathbf{x}) \neq 0; \text{ otherwise } \mathbf{x} \text{ is } \textit{singular}, \tag{33}$$

$$\mathbf{x}^{1/2} \stackrel{\text{def}}{=} \lambda_1^{1/2}\mathbf{c_1} + \lambda_2^{1/2}\mathbf{c_2} \quad \text{for } \mathbf{x} \succcurlyeq_{\mathcal{Q}} \mathbf{0}, \tag{34}$$

$$f(\mathbf{x}) \stackrel{\text{def}}{=} f(\lambda_1)\mathbf{c_1} + f(\lambda_2)\mathbf{c_2} \quad \text{for } f \text{ continuous at } \lambda_1 \text{ and } \lambda_2. \tag{35}$$

Note that $\mathbf{x} \circ \mathbf{x}^{-1} = \mathbf{e}$ and $\mathbf{x}^{1/2} \circ \mathbf{x}^{1/2} = \mathbf{x}$.

In addition to $\operatorname{Arw}(\mathbf{x})$, there is another important linear transformation associated with each $\mathbf{x}$ called the *quadratic representation* that is defined as

$$Q_{\mathbf{x}} \stackrel{\text{def}}{=} 2\operatorname{Arw}^2(\mathbf{x}) - \operatorname{Arw}\left(\mathbf{x}^2\right) = \begin{pmatrix} \|\mathbf{x}\|^2 & 2x_0\bar{\mathbf{x}}^\top \\ 2x_0\bar{\mathbf{x}} & \det(\mathbf{x})I + 2\bar{\mathbf{x}}\bar{\mathbf{x}}^\top \end{pmatrix} = \left(2\mathbf{x}\mathbf{x}^\top - \det(\mathbf{x})R\right). \tag{36}$$

Thus $Q_{\mathbf{x}}\mathbf{y} = 2\mathbf{x} \circ (\mathbf{x} \circ \mathbf{y}) - \mathbf{x}^2 \circ \mathbf{y} = 2(\mathbf{x}^\top\mathbf{y})\mathbf{x} - \det(\mathbf{x})R\mathbf{y}$ is a vector of quadratic functions in $\mathbf{x}$. This operator may seem somewhat arbitrary, but it is of fundamental importance. It is analogous to the operator that sends $Y$ to $XYX$ in the algebra of symmetric matrices and retains many of the latter's properties.

The most prominent properties of both $\mathrm{Arw}(\mathbf{x})$ and $Q_{\mathbf{x}}$ can be obtained easily from studying their eigenvalue/eigenvector structure. We remind the reader that in matrix algebra two square matrices $X$ and $Y$ commute, that is $XY = YX$, if and only if they share a common system of eigenvectors.

**Theorem 3** *Let $\mathbf{x}$ have the spectral decomposition $\mathbf{x} = \lambda_1\mathbf{c_1} + \lambda_2\mathbf{c_2}$. Then,*

1. $\mathrm{Arw}(\mathbf{x})$ *and $Q_{\mathbf{x}}$ commute and thus share a system of eigenvectors.*
2. $\lambda_1 = x_0 + \|\bar{\mathbf{x}}\|$ *and* $\lambda_2 = x_0 - \|\bar{\mathbf{x}}\|$ *are eigenvalues of* $\mathrm{Arw}(\mathbf{x})$. *Furthermore, if $\lambda_1 \neq \lambda_2$ then each one has multiplicity one; the corresponding eigenvectors are $\mathbf{c_1}$ and $\mathbf{c_2}$, respectively. Furthermore, $x_0$ is an eigenvalue of* $\mathrm{Arw}(\mathbf{x})$, *and has a multiplicity of $n-2$ when $\mathbf{x} \neq \mathbf{0}$.*
3. $\lambda_1^2 = (x_0 + \|\bar{\mathbf{x}}\|)^2$ *and* $\lambda_2^2 = (x_0 - \|\bar{\mathbf{x}}\|)^2$ *are eigenvalues of $Q_{\mathbf{x}}$. Furthermore, if $\lambda_1 \neq \lambda_2$ then each one has multiplicity one; the corresponding eigenvectors are $\mathbf{c_1}$ and $\mathbf{c_2}$, respectively. In addition,* $\det(\mathbf{x}) = x_0^2 - \|\bar{\mathbf{x}}\|^2$ *is an eigenvalue of $Q_{\mathbf{x}}$, and has a multiplicity of $n-2$ when $\mathbf{x}$ is nonsingular and $\lambda_1 \neq \lambda_2$.*

**Proof:** The statement that $\mathrm{Arw}(\mathbf{x})$ and $Q_{\mathbf{x}}$ commute is a direct consequence of the definition of $Q_{\mathbf{x}}$ as $2\mathrm{Arw}^2(\mathbf{x}) - \mathrm{Arw}\left(\mathbf{x}^2\right)$, and the fact that $\mathrm{Arw}(\mathbf{x})$ and $\mathrm{Arw}\left(\mathbf{x}^2\right)$ commute. Parts (2) and (3) can be proved by inspection. ∎

**Corollary 4** *For each $\mathbf{x}$ we have*

1. *$Q_{\mathbf{x}}$ is nonsingular if and only if $\mathbf{x}$ is nonsingular,*
2. *if $\mathbf{x} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$, $\mathbf{x}$ is nonsingular if and only if* $\mathrm{Arw}(\mathbf{x})$ *is nonsingular.*

By definition $\circ$ is commutative. However we need to capture the analog of commutativity in matrices. Therefore, we define commutativity in terms of sharing a common Jordan frame.

**Definition 5** *The elements $\mathbf{x}$ and $\mathbf{y}$* operator commute *if they share a Jordan frame, that is*

$$\mathbf{x} = \lambda_1\mathbf{c_1} + \lambda_2\mathbf{c_2} \qquad \mathbf{y} = \omega_1\mathbf{c_1} + \omega_2\mathbf{c_2}$$

*for a Jordan frame $\{\mathbf{c_1}, \mathbf{c_2}\}$.*

**Theorem 6** *The following statements are equivalent:*

1. *$\mathbf{x}$ and $\mathbf{y}$ operator commute.*
2. $\mathrm{Arw}(\mathbf{x})$ *and* $\mathrm{Arw}(\mathbf{y})$ *commute. Therefore, for all $\mathbf{z} \in \Re^n$, $\mathbf{x} \circ (\mathbf{y} \circ \mathbf{z}) = \mathbf{y} \circ (\mathbf{x} \circ \mathbf{z})$.*
3. *$Q_{\mathbf{x}}$ and $Q_{\mathbf{y}}$ commute.*

**Proof:** Immediate from the eigenvalue structure of $\operatorname{Arw}(\mathbf{x})$ and $Q_{\mathbf{x}}$ stated in Theorem 3. ∎

**Corollary 7** *The vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *operator commute if and only if* $\bar{\mathbf{x}} = \mathbf{0}$ *or* $\bar{\mathbf{y}} = \mathbf{0}$ *or* $\bar{\mathbf{x}}$ *and* $\bar{\mathbf{y}}$ *are proportional, that is* $\bar{\mathbf{x}} = \alpha\bar{\mathbf{y}}$ *for some real number* $\alpha$.

Therefore, $\mathbf{x}$ and $\mathbf{y}$ commute if either one of them is a multiple of the identity, or else their projections on the hyperplane $x_0 = 0$ are collinear. Now we list some properties of $Q_{\mathbf{x}}$:

**Theorem 8** *For each* $\mathbf{x}, \mathbf{y}, \mathbf{u} \in \Re^n$, $\mathbf{x}$ *nonsingular,* $\alpha \in \Re$, *and integer* $t$ *we have*

1. $Q_{\alpha\mathbf{y}} = \alpha^2 Q_{\mathbf{y}}$.
2. $Q_{\mathbf{x}}\mathbf{x}^{-1} = \mathbf{x}$ *and thus* $Q_{\mathbf{x}}^{-1}\mathbf{x} = \mathbf{x}^{-1}$
3. $Q_{\mathbf{y}}\mathbf{e} = \mathbf{y}^2$.
4. $Q_{\mathbf{x}^{-1}} = Q_{\mathbf{x}}^{-1}$ *and more generally* $Q_{\mathbf{x}^t} = Q_{\mathbf{x}}^t$.
5. *If* $y$ *is nonsingular,* $(Q_{\mathbf{x}}\mathbf{y})^{-1} = Q_{\mathbf{x}^{-1}}\mathbf{y}^{-1}$, *and more generally if* $\mathbf{x}$ *and* $\mathbf{y}$ *operator commute then* $(Q_{\mathbf{x}}\mathbf{y})^t = Q_{\mathbf{x}^t}\mathbf{y}^t$.
6. *The gradient* $\nabla_{\mathbf{x}}\big(\ln\det(\mathbf{x})\big) = 2\mathbf{x}^{-1}$, *and the Hessian* $\nabla^2_{\mathbf{x}}\big(\ln\det(\mathbf{x})\big) = -2Q_{\mathbf{x}^{-1}}$, *and in general,* $D_{\mathbf{u}}\mathbf{x}^{-1} = -2Q_{\mathbf{x}^{-1}}\mathbf{u}$, *where* $D_{\mathbf{u}}$ *is the differential in the direction of* $\mathbf{u}$.
7. $Q_{Q_{\mathbf{y}}\mathbf{x}} = Q_{\mathbf{y}}Q_{\mathbf{x}}Q_{\mathbf{y}}$
8. $\det\big(Q_{\mathbf{x}}\mathbf{y}\big) = \det^2(\mathbf{x})\det(\mathbf{y})$.

**Proof:** (1) is trivial. Note that $\mathbf{x}^{-1} = R\mathbf{x}/\det(\mathbf{x})$ and

$$Q_{\mathbf{x}^{-1}} = \frac{1}{\det^2(\mathbf{x})}\begin{pmatrix} \|\mathbf{x}\|^2 & -2x_0\bar{\mathbf{x}}^\top \\ -2x_0\bar{\mathbf{x}} & \det(\mathbf{x})I + 2\bar{\mathbf{x}}\bar{\mathbf{x}}^\top \end{pmatrix} = \frac{1}{\det^2(\mathbf{x})}RQ_{\mathbf{x}}R.$$

With these observations (2), (3), first part of (4), (5), and (7) can be proved by simple algebraic manipulations. The second part of (4) as well as (5) are proved by simple inductions. For (6) it suffices to show that the Jacobian of the vector field $\mathbf{x} \to \mathbf{x}^{-1}$ is $-Q_{\mathbf{x}^{-1}}$ which again can be verified by inspection. We give a proof of (8) as a representative. Let $\mathbf{z} = Q_{\mathbf{x}}\mathbf{y}$, $\alpha = \mathbf{x}^\top\mathbf{y}$ and $\gamma = \det(\mathbf{x})$. Then $\mathbf{z} = 2\alpha\mathbf{x} - \gamma R\mathbf{y}$ and we have

$$\begin{aligned} z_0^2 &= \big(2\alpha x_0 - \gamma y_0\big)^2 = 4\alpha^2 x_0^2 - 4\alpha\gamma x_0 y_0 + \gamma^2 y_0^2 \\ \|\bar{\mathbf{z}}\|^2 &= \|2\alpha\bar{\mathbf{x}} + \gamma\bar{\mathbf{y}}\|^2 = 4\alpha^2\|\bar{\mathbf{x}}\|^2 + 4\alpha\gamma\bar{\mathbf{x}}^\top\bar{\mathbf{y}} + \gamma^2\|\bar{\mathbf{y}}\|^2. \end{aligned}$$

Hence we have

$$\begin{aligned} \det(\mathbf{z}) &= z_0^2 - \|\bar{\mathbf{z}}\|^2 = 4\alpha^2\big(x_0^2 - \|\bar{\mathbf{x}}\|^2\big) - 4\alpha\gamma(x_0y_0 + \bar{\mathbf{x}}^\top\bar{\mathbf{y}}) + \gamma^2\big(y_0^2 - \|\bar{\mathbf{y}}\|^2\big) \\ &= \gamma^2\big(y_0^2 - \|\bar{\mathbf{y}}\|^2\big) = \det(\mathbf{x})^2\det(\mathbf{y}). \end{aligned}$$

■

**Theorem 9** *Let* $\mathbf{p} \in \Re^n$ *be nonsingular. Then* $Q_{\mathbf{p}}(\mathcal{Q}) = \mathcal{Q}$*; likewise,* $Q_{\mathbf{p}}(\operatorname{int} \mathcal{Q}) = \operatorname{int} \mathcal{Q}$.

**Proof:** Let $\mathbf{x} \in \mathcal{Q}$ (respectively, $\mathbf{x} \in \operatorname{int} \mathcal{Q}$) and $\mathbf{y} = Q_{\mathbf{p}}\mathbf{x}$. By part 8 of Theorem 8 $\det(\mathbf{y}) \geq 0$ (respectively, $\det(\mathbf{y}) > 0$). Therefore, either $\mathbf{y} \in \mathcal{Q}$ or $\mathbf{y} \in -\mathcal{Q}$ (respectively, $\mathbf{y} \in -\operatorname{int} \mathcal{Q}$ or $\mathbf{y} \in \operatorname{int} \mathcal{Q}$). Since $2y_0 = \operatorname{tr} \mathbf{y}$, it suffices to show that $y_0 \geq 0$. By using the fact that $x_0 \geq \|\bar{\mathbf{x}}\|$ and then applying the Cauchy-Schwarz inequality to $|\mathbf{p}^\top \mathbf{x}|$ we get

$$\begin{aligned}
y_0 &= 2\big(p_0 x_0 + \bar{\mathbf{p}}^\top \bar{\mathbf{x}}\big)p_0 - \big(p_0^2 - \|\bar{\mathbf{p}}\|^2\big)x_0 = x_0 p_0^2 + x_0\|\bar{\mathbf{p}}\|^2 + 2p_0(\bar{\mathbf{p}}^\top \bar{\mathbf{x}}) \\
&\geq \|\bar{\mathbf{x}}\|\big(p_0^2 + \|\bar{\mathbf{p}}\|^2\big) + 2p_0(\bar{\mathbf{p}}^\top \bar{\mathbf{x}}) \\
&\geq \|\bar{\mathbf{x}}\|\big(p_0^2 + \|\bar{\mathbf{p}}\|^2\big) - 2\big|p_0\big|\,\|\bar{\mathbf{p}}\|\,\|\bar{\mathbf{x}}\| \\
&= \|\bar{\mathbf{x}}\|\big(|p_0| - \|\bar{\mathbf{p}}\|\big)^2 \geq 0.
\end{aligned}$$

Thus $Q_{\mathbf{p}}(\mathcal{Q}) \subseteq \mathcal{Q}$ (respectively, $Q_{\mathbf{p}}(\operatorname{int} \mathcal{Q}) \subseteq Q_{\mathbf{p}}(\operatorname{int} \mathcal{Q})$). Also $\mathbf{p}^{-1}$ is invertible, therefore $Q_{\mathbf{p}^{-1}}\mathbf{x} \in \mathcal{Q}$ for each $\mathbf{x} \in \mathcal{Q}$. It follows that for every $\mathbf{x} \in \mathcal{Q}$, since $\mathbf{x} = Q_{\mathbf{p}}Q_{\mathbf{p}^{-1}}\mathbf{x}$, $\mathbf{x}$ is the image of $Q_{\mathbf{p}^{-1}}\mathbf{x}$; that is $\mathcal{Q} \subseteq Q_{\mathbf{p}}(\mathcal{Q})$; similarly $\operatorname{int} \mathcal{Q} \subseteq Q_{\mathbf{p}}(\operatorname{int} \mathcal{Q})$. ■

We will prove a theorem which will be used later in the discussion of interior point methods.

**Theorem 10** *Let* $\mathbf{x}$*,* $\mathbf{z}$*,* $\mathbf{p}$ *be all positive definite. Define* $\widetilde{\mathbf{x}} = Q_{\mathbf{p}}\mathbf{x}$ *and* $\underset{\sim}{\mathbf{z}} = Q_{\mathbf{p}^{-1}}\mathbf{z}$*. Then*

1. *the vectors* $\mathbf{a} = Q_{\mathbf{x}^{1/2}}\mathbf{z}$ *and* $\mathbf{b} = Q_{\widetilde{\mathbf{x}}^{1/2}}\underset{\sim}{\mathbf{z}}$ *have the same spectrum (that is the same set of eigenvalues with the same multiplicities);*

2. *the vectors* $Q_{\mathbf{x}^{1/2}}\mathbf{z}$ *and* $Q_{\mathbf{z}^{1/2}}\mathbf{x}$ *have the same spectrum.*

**Proof:** By part 3 of Theorem 3 the eigenvalues of $Q_{\mathbf{a}}$ and $Q_{\mathbf{b}}$ are pairwise products of eigenvalues of $\mathbf{a}$ and $\mathbf{b}$, respectively. Since by Theorem 9 the eigenvalues of $\mathbf{a}$ and $\mathbf{b}$ are all positive, $\mathbf{a}$ and $\mathbf{b}$ have the same eigenvalues if and only if their quadratic representations are similar matrices.

1. By part 7 of Theorem 8,

$$Q_{\mathbf{a}} = Q_{Q_{\mathbf{x}^{1/2}}\mathbf{z}} = Q_{\mathbf{x}^{1/2}}Q_{\mathbf{z}}Q_{\mathbf{x}^{1/2}}$$

and

$$Q_{\mathbf{b}} = Q_{Q_{\widetilde{\mathbf{x}}^{1/2}}\underset{\sim}{\mathbf{z}}} = Q_{\widetilde{\mathbf{x}}^{1/2}}Q_{\underset{\sim}{\mathbf{z}}}Q_{\widetilde{\mathbf{x}}^{1/2}}.$$

Since $Q_{\mathbf{x}}^t = Q_{\mathbf{x}^t}$ (part 4 of Theorem 8), $Q_{\mathbf{a}}$ is similar to $Q_{\mathbf{x}}Q_{\mathbf{z}}$ and $Q_{\mathbf{b}}$ is similar to $Q_{\widetilde{\mathbf{x}}}Q_{\underset{\sim}{\mathbf{z}}}$. From the definition of $\widetilde{\mathbf{x}}$ and $\underset{\sim}{\mathbf{z}}$ we obtain

$$Q_{\widetilde{\mathbf{x}}}Q_{\underset{\sim}{\mathbf{z}}} = Q_{\mathbf{p}}Q_{\mathbf{x}}Q_{\mathbf{p}}Q_{\mathbf{p}^{-1}}Q_{\mathbf{z}}Q_{p^{-1}} = Q_{\mathbf{p}}Q_{\mathbf{x}}Q_{\mathbf{z}}Q_{\mathbf{p}}^{-1}.$$

Therefore, $Q_{\mathbf{a}}$ and $Q_{\mathbf{b}}$ are similar, which proves part 1.

2. $Q_{Q_{\mathbf{x}^{1/2}}}\mathbf{z} = Q_{\mathbf{x}^{1/2}}Q_{\mathbf{z}}Q_{\mathbf{x}^{1/2}}$ and $Q_{\mathbf{x}}Q_{\mathbf{z}}$ are similar; so are $Q_{Q_{\mathbf{z}^{1/2}}}\mathbf{x} = Q_{\mathbf{z}^{1/2}}Q_{\mathbf{x}}Q_{\mathbf{z}^{1/2}}$ and $Q_{\mathbf{z}}Q_{\mathbf{x}}$. But $Q_{\mathbf{x}}Q_{\mathbf{z}}$ and $Q_{\mathbf{z}}Q_{\mathbf{x}}$ are also similar which proves part 2 of the theorem.

■

The algebraic structure we have studied thus far is called *simple* for $n \geq 3$. This means that it cannot be expressed as direct sum of other like algebras. For SOCP, as alluded to in the introduction section, we generally deal with vectors that are partitioned into sub-vectors, each of which belongs to the second-order cone in its subspace. This implies that the underlying algebra is not simple but rather is a direct sum of algebras.

**Definition 11** *Let* $\mathbf{x} = (\mathbf{x_1}; \ldots; \mathbf{x_r})$, $\mathbf{y} = (\mathbf{y_1}; \ldots; \mathbf{y_r})$, *and* $\mathbf{x_i}, \mathbf{y_i} \in \Re^{n_i}$ *for* $i = 1, \ldots, r$. *Then*

1. $\mathbf{x} \circ \mathbf{y} \stackrel{\text{def}}{=} (\mathbf{x_1} \circ \mathbf{y_1}; \ldots; \mathbf{x_r} \circ \mathbf{y_r})$.
2. $\operatorname{Arw}(\mathbf{x}) \stackrel{\text{def}}{=} \operatorname{Arw}(\mathbf{x_1}) \oplus \cdots \oplus \operatorname{Arw}(\mathbf{x_r})$.
3. $Q_{\mathbf{x}} \stackrel{\text{def}}{=} Q_{\mathbf{x_1}} \oplus \cdots \oplus Q_{\mathbf{x_r}}$.
4. *The characteristic polynomial of* $\mathbf{x}$ *is* $p(\mathbf{x}) \stackrel{\text{def}}{=} \prod_i p_i(\mathbf{x_i})$.
5. $\mathbf{x}$ *has* $2r$ *eigenvalues (including multiplicities) comprised of the union of the eigenvalues of* $\mathbf{x_i}$. *Also* $2r$ *is called the* rank *of* $\mathbf{x}$.
6. *The cone of squares* $\mathcal{Q} \stackrel{\text{def}}{=} \mathcal{Q}_{n_1} \times \cdots \times \mathcal{Q}_{n_r}$.
7. $\|\mathbf{x}\|_F^2 \stackrel{\text{def}}{=} \sum_i \|\mathbf{x_i}\|_F^2$.
8. $\|\mathbf{x}\|_2 \stackrel{\text{def}}{=} \max_i \|\mathbf{x_i}\|_2$.
9. $\mathbf{x}^{-1} \stackrel{\text{def}}{=} (\mathbf{x_1}^{-1}; \cdots; \mathbf{x_r}^{-1})$ *and more generally for all* $t \in \Re$, $\mathbf{x}^t \stackrel{\text{def}}{=} (\mathbf{x_1}^t; \cdots; \mathbf{x_r}^t)$, *when all* $\mathbf{x_i}^t$ *are well defined.*
10. $\mathbf{x}$ *and* $\mathbf{y}$ *operator commute if and only if* $\mathbf{x_i}$ *and* $\mathbf{y_i}$ *operator commute for all* $i = 1, \ldots, r$.

Note that Corollary 4 and Theorems 6, 8, 9 and 10, and Corollary 7 extend to the direct sum algebra verbatim.

# 5 Duality and Complementarity

Since SOCPs are convex programming problems, a duality theory for them can be developed. While much of this theory is very similar to duality theory for LP, there are many ways in which the theory for SOCP differs from that for LP.

We now consider these similarities and differences for the standard form SOCP and its dual (2).

As in LP, we have from

$$\mathbf{c}^\top\mathbf{x} - \mathbf{b}^\top\mathbf{y} = (\mathbf{y}^\top A + \mathbf{z}^\top)\mathbf{x} - \mathbf{x}^\top A^\top\mathbf{y} = \mathbf{x}^\top\mathbf{z}$$

and from the self-duality (self-polarity) of the cone $\mathcal{Q}$ the following

**Lemma 12 (Weak Duality Lemma)** *If $\mathbf{x}$ is any primal feasible solution and $(\mathbf{y},\mathbf{z})$ is any dual feasible solution, then the duality gap $\mathbf{c}^\top\mathbf{x} - \mathbf{b}^\top\mathbf{y} = \mathbf{z}^\top\mathbf{x} \geq 0$.*

In the SOCP case we also have the following

**Theorem 13 (Strong Duality Theorem)** *If the primal and dual problems in (2) have strictly feasible solutions ($\mathbf{x_i} \succ_{\mathcal{Q}} \mathbf{0}$, $\mathbf{z_i} \succ_{\mathcal{Q}} \mathbf{0}, \forall i$), then both have optimal solutions $\mathbf{x}^*, (\mathbf{y}^*, \mathbf{z}^*)$ and $p^* \stackrel{\text{def}}{=} \mathbf{c}^\top\mathbf{x}^* = d^* \stackrel{\text{def}}{=} \mathbf{b}^\top\mathbf{y}^*$, (i.e., ${\mathbf{x}^*}^\top\mathbf{z}^* = 0$).*

This theorem and Theorem 14 below are in fact true for all cone-LPs over any closed, pointed, convex and full-dimensional cone $\mathcal{K}$, and can be proved by using a general form of Farkas' lemma (e.g. see [NN94]).

Theorem 13 is weaker than the result that one has in the LP case. In LP, for the strong duality conclusion to hold one only needs that either (i) both (P) and (D) have feasible solutions, or (ii) (P) has feasible solutions and the objective value is bounded below in the feasible region. For SOCP neither (i) nor (ii) is enough to guarantee the strong duality result. To illustrate this, consider the following primal-dual pair of problems:

$$\begin{array}{llll} (\mathrm{P}) & p^* = & \inf & (1,-1,0)\mathbf{x} \\ & & \text{s.t.} & (0,0,1)\mathbf{x} = 1 \\ & & & \mathbf{x} \succcurlyeq_{\mathcal{Q}} \mathbf{0} \end{array} \qquad \begin{array}{llll} (\mathrm{D}) & d^* = & \sup & y \\ & & \text{s.t.} & \begin{pmatrix}0\\0\\1\end{pmatrix} y + \mathbf{z} = \begin{pmatrix}1\\-1\\0\end{pmatrix} \\ & & & \mathbf{z} \succcurlyeq_{\mathcal{Q}} \mathbf{0} \end{array} .$$

The linear constraints in problem (P) require that $x_2 = 1$; hence problem (P) is equivalent to the problem:

$$\begin{array}{lll} p^* = & \inf & x_0 - x_1 \\ & \text{s.t.} & x_0 \geq \sqrt{x_1^2 + 1}. \end{array}$$

Since $x_0 - x_1 > 0$ and $x_0 - x_1 \to 0$ as $x_0 = \sqrt{x_1^2+1} \to \infty$, $p^* = \inf(x_0 - x_1) = 0$; i.e., problem (P) has no finite optimal solution, but the objective function over the feasible region is bounded below by 0 .

The linear constraints in problem (D) require that $z^\top = (1,-1,-y)$; hence problem (D) is equivalent to the problem:

$$\begin{array}{rll} d^* = & \sup & y \\ & \text{s.t.} & 1 \geq \sqrt{1+y^2}. \end{array}$$

$y = 0$ is the only feasible solution, and hence, it is the unique optimal solution to this problem and $d^* = 0$.

The above example illustrates the following

**Theorem 14 (Semi-strong Duality Theorem)** *If (P) has strictly feasible point* $\hat{\mathbf{x}}$ *(i.e.* $\hat{\mathbf{x}} \in \operatorname{int}(\mathcal{Q})$*) and its objective is bounded below over the feasible region, then (D) is solvable and* $p^* = d^*$*.*

In LP, the complementary slackness conditions follow from the fact that if $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{z} \geq \mathbf{0}$, then the duality gap $\mathbf{x}^\top \mathbf{z} = 0$ if and only if $x_i z_i = 0$, for all $i$. In SOCP, since the second-order cone $\mathcal{Q}$ is self-dual, it is clear geometrically that $\mathbf{x}^\top \mathbf{z} = 0$, i.e., $\mathbf{x} \in \mathcal{Q}$ is orthogonal to $\mathbf{z} \in \mathcal{Q}$, if and only if for each block either $\mathbf{x_i}$ or $\mathbf{z_i}$ is zero, or else they lie on opposite sides of the boundary of $\mathcal{Q}_{n_i}$, i.e., $\mathbf{x_i}$ is the reflection of $\mathbf{z_i}$ through the axis corresponding to the $0^{\text{th}}$ coordinate in addition to $\mathbf{x_i}$ being orthogonal to $\mathbf{z_i}$. (More succinctly $\mathbf{x_i} = \alpha_i R \mathbf{z_i}$, for some $\alpha_i > 0$.) This result is given by the following

**Lemma 15 (Complementarity Conditions)** *Suppose that* $\mathbf{x} \in \mathcal{Q}$ *and* $\mathbf{z} \in \mathcal{Q}$*, (i.e.,* $\mathbf{x_i} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$ *and* $\mathbf{z_i} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$*, for all* $i = 1, \ldots, r$*). Then* $\mathbf{x}^\top \mathbf{z} = 0$ *if, and only if,* $\mathbf{x_i} \circ \mathbf{z_i} = \operatorname{Arw}(\mathbf{x_i}) \operatorname{Arw}(\mathbf{z_i}) \mathbf{e} = \mathbf{0}$ *for* $i = 1, \ldots, r$*, or equivalently,*

*(i)* $\mathbf{x_i}^\top \mathbf{z_i} = x_{i0} z_{i0} + \overline{\mathbf{x}}_\mathbf{i}^\top \overline{\mathbf{z}}_\mathbf{i} = 0, \quad i = 1, \ldots, r$

*(ii)* $x_{i0} \overline{\mathbf{z}}_\mathbf{i} + z_{i0} \overline{\mathbf{x}}_\mathbf{i} = 0, \quad i = 1, \ldots, r.$

**Proof:** If $x_{i0} = 0$ or $z_{i0} = 0$ the result is trivial, since $\mathbf{x_i} = \mathbf{0}$ in the first case and $\mathbf{z_i} = \mathbf{0}$ in the second. Therefore, we need only consider the case where ($x_{i0} > 0$ and $z_{i0} > 0$). By the Cauchy-Schwarz inequality and the assumption that $\mathbf{x_i} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$ and $\mathbf{z_i} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$,

$$\overline{\mathbf{x}}_\mathbf{i}^\top \overline{\mathbf{z}}_\mathbf{i} \geq -\|\overline{\mathbf{x}}_\mathbf{i}\| \cdot \|\overline{\mathbf{z}}_\mathbf{i}\| \geq -x_{i0} z_{i0}. \tag{37}$$

Therefore, $\mathbf{x_i}^\top \mathbf{z_i} = x_{i0} z_{i0} + \overline{\mathbf{x}}_\mathbf{i}^\top \overline{\mathbf{z}}_\mathbf{i} \geq 0$, and $\mathbf{x}^\top \mathbf{z} = \sum_{i=1}^k \mathbf{x_i}^\top \mathbf{z_i}^\top = 0$ if and only if $\mathbf{x_i}^\top \mathbf{z_i} = 0$, for $i = 1, \ldots, k$. Now $\mathbf{x_i}^\top \mathbf{z_i} = x_{i0} z_{i0} + \overline{\mathbf{x}}_\mathbf{i}^\top \overline{\mathbf{z}}_\mathbf{i} = 0$ if and only if $\overline{\mathbf{x}}_\mathbf{i}^\top \overline{\mathbf{z}}_\mathbf{i} = -x_{i0} z_{i0}$, hence if and only if the inequalities in (37) are satisfied as equalities. But this is true if and only if either $\mathbf{x_i} = \mathbf{0}$ or $\mathbf{z_i} = \mathbf{0}$, in which case (i) and (ii) hold trivially, or $\mathbf{x_i} \neq \mathbf{0}$ and $\mathbf{z_i} \neq \mathbf{0}$, $\overline{\mathbf{x}}_\mathbf{i} = -\alpha_i \overline{\mathbf{z}}_\mathbf{i}$, where $\alpha > 0$, and $x_{i0} = \|\overline{\mathbf{x}}_\mathbf{i}\| = \alpha \|\overline{\mathbf{z}}_\mathbf{i}\| = \alpha z_{i0}$, i.e., $\overline{\mathbf{x}}_\mathbf{i} + \frac{x_{i0}}{z_{i0}} \overline{\mathbf{z}}_\mathbf{i} = \mathbf{0}$. ■

Combining primal and dual feasibility with the above complementarity conditions yields optimality conditions for a primal-dual pair of SOCPs. Specifically, we have the following

**Theorem 16 (Optimality Conditions)** *If (P) and (D) have strictly feasible solutions (i.e., there is a feasible solution pair* $(\hat{\mathbf{x}}, (\hat{\mathbf{y}}, \hat{\mathbf{z}}))$ *such that* $\hat{\mathbf{x}}_{\mathbf{i}} \succ_{\mathcal{Q}} \mathbf{0}$, $\hat{\mathbf{z}}_{\mathbf{i}} \succ_{\mathcal{Q}} \mathbf{0}, \forall i$*), then* $(\mathbf{x}, (\mathbf{y}, \mathbf{z}))$ *is an optimal solution pair for these problems if and only if*

$$\begin{aligned} &A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in \mathcal{Q}, \\ &A^\top \mathbf{y} + \mathbf{z} = \mathbf{c}, \quad \mathbf{z} \in \mathcal{Q}, \\ &\mathbf{x} \circ \mathbf{z} = \mathbf{0}. \end{aligned} \tag{38}$$

When $r = 1$, that is when there is only a single block, the system of equations in (38) can be solved analytically. If $\mathbf{b} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ and any dual feasible $(\mathbf{y}, \mathbf{z})$ yields an optimal primal-dual solution pair. If $\mathbf{c} \in \operatorname{Span}(A^\top)$, that is $A^\top \mathbf{y} = \mathbf{c}$ for some $\mathbf{y}$, then $\mathbf{z} = \mathbf{0}$ and that $\mathbf{y}$ and any primal feasible $\mathbf{x}$ yields an optimal primal-dual solution pair. Otherwise, $\mathbf{x}$ and $\mathbf{z}$ are nonzero in any optimal solution pair, and by the last equation of (38), $\mathbf{z} = \alpha R \mathbf{x}$ where $\alpha = z_0/x_0 > 0$. Since we are assuming that $A = (\mathbf{a}, \overline{A})$ has full row rank, there are two cases to consider.

**Case 1:** $ARA^T$ **is nonsingular.**

Substituting $\alpha R\mathbf{x}$ for $\mathbf{z}$ in the second equation of (38) and solving the resulting system of equations for $\mathbf{x}$ and $\mathbf{y}$ gives

$$\mathbf{y} = \left(ARA^T\right)^{-1}(AR\mathbf{c} - \alpha \mathbf{b}), \tag{39}$$

$$\mathbf{x} = \tfrac{1}{\alpha} P_R \mathbf{c} + RA^\top \left(ARA^\top\right)^{-1} \mathbf{b} \tag{40}$$

where $P_R = R - RA^\top (ARA^\top)^{-1} AR$. Finally using the fact that $0 = \mathbf{z}^\top \mathbf{x} = \alpha \mathbf{x}^T R \mathbf{x}$, we obtain

$$\alpha = \sqrt{\frac{-\mathbf{c}^\top P_R \mathbf{c}}{\mathbf{b}^T \left(ARA^\top\right)^{-1} \mathbf{b}}}. \tag{41}$$

If we let $A^+ = A^\top \left(AA^\top\right)^{-1}$ be the pseudoinverse of $A$, and let $P$ denote the orthogonal projection operator $P = I - A^+ A$ that orthogonally projects all vectors in $\Re^n$ onto the kernel of $A$, we can express $\left(ARA^\top\right)^{-1}$ and $P_R$ as

$$\begin{aligned} \left(ARA^\top\right)^{-1} &= -\left(AA^T\right)^{-1} - \tfrac{2}{\gamma}\left(AA^\top\right)^{-1} \mathbf{a}\mathbf{a}^\top \left(AA^\top\right)^{-1} \\ P_R &= R\left(P - \tfrac{2}{\gamma}\left(\left(A^+\mathbf{a}\right)\left(A^+\mathbf{a}\right)^\top - A^+\mathbf{a}\mathbf{e}^\top\right)\right), \end{aligned}$$

where $\gamma = 1 - 2\mathbf{a}^\top \left(AA^\top\right)^{-1} \mathbf{a}$; note that $\gamma \neq 0$ if and only if $ARA^\top$ is nonsingular. By substituting these expressions for $\left(ARA^\top\right)^{-1}$ and $P_R$ into (39)–(41) and further algebraically manipulating these results we obtain

$$\mathbf{y} = \left(AA^\top\right)^{-1}(A\mathbf{c} + \alpha \mathbf{b} - \delta \mathbf{a}) \tag{42}$$

$$\alpha R \mathbf{x} = \mathbf{z} = \mathbf{c} - A^\top \mathbf{y} = P\mathbf{c} - A^+(\alpha \mathbf{b} - \delta \mathbf{a}) \tag{43}$$

where

$$\alpha = \sqrt{\frac{-\gamma \mathbf{c}^\top P\mathbf{c} + 2\left(\mathbf{e}^\top P\mathbf{c}\right)^2}{\gamma \mathbf{b}^\top \left(AA^\top\right)^{-1}\mathbf{b} + 2\left(\mathbf{a}^\top \left(AA^\top\right)^{-1}\mathbf{b}\right)^2}} \quad \text{and} \tag{44}$$

$$\delta = \frac{\mathbf{c}^\top P\mathbf{c} + \alpha^2 \mathbf{b}^\top \left(AA^\top\right)^{-1}\mathbf{b}}{\mathbf{e}^\top P\mathbf{c} + \alpha \mathbf{a}^\top \left(AA^\top\right)^{-1}\mathbf{b}} \tag{45}$$

**Case 2: $ARA^\top$ is singular.**

In this case $\gamma = 1 - 2\mathbf{a}^\top \left(AA^\top\right)^{-1}\mathbf{a} = 0$ and $ARA^\top$ has a 1-dimensional kernel $\mathcal{N}$. Note that for $\mathbf{v} \in \mathcal{N}$ and $\mathbf{v} \neq \mathbf{0}$, $\mathbf{u} = RA^\top\mathbf{v} \neq \mathbf{0}$, $A\mathbf{u} = \mathbf{0}$ and $\mathbf{u}^\top R\mathbf{u} = \mathbf{v}^\top A\mathbf{u} = 0$. Therefore by appropriately choosing the sign of $\mathbf{v}$, the corresponding $\mathbf{u}$ lies in both the kernel of $A$ and the boundary of $\mathcal{Q}$.

As above, setting $\mathbf{z} = \alpha R\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ satisfy (42) and (43); but since

$$c_0 = \mathbf{a}^\top\mathbf{y} + z_0 = \mathbf{a}^\top \left(AA^\top\right)^{-1}(A\mathbf{c} + \alpha\mathbf{b}) + z_0\gamma$$

and $\gamma = 0$,

$$\alpha = \frac{\mathbf{e}^\top P\mathbf{c}}{\mathbf{a}^\top \left(AA^\top\right)^{-1}\mathbf{b}}. \tag{46}$$

After some algebra it can be shown from $\mathbf{z}^\top R\mathbf{z} = 0$ that

$$\delta = \frac{\mathbf{c}^\top P\mathbf{c} + \alpha^2 \mathbf{b}^\top \left(AA^\top\right)^{-1}\mathbf{b}}{2\mathbf{e}^\top P\mathbf{c}}. \tag{47}$$

But it is clear that (45) is identical to (47) where $\gamma = 0$, i.e. $\alpha$ is given by (46). Thus formulas (42)–(45) can also be used when $ARA^\top$ is singular.

# 6 Nondegeneracy and Strict Complementarity

The concepts of primal and dual nondegeneracy and strict complementarity have been extended to all cone-LP's. In the case of semidefinite programming Alizadeh et al. [AHO97] gave concrete characterizations which involved checking the rank of certain matrices derived from the data. Their approach was extended by Faybusovich [Fay97b] to symmetric cones. Finally Pataki [Pat96] gave an alternative, but essentially equivalent formulation of primal and dual nondegeneracy and strict complementarity for all cone-LP's and also analyzed the special case of QCQP problems.

In this section we discuss these concepts in the case of SOCPs. Notice that in the primal standard form (2) the feasible region is the intersection of $\mathcal{Q}$ and an affine set $\mathcal{A} = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$. Let $\mathbf{x}$ be a feasible point. The notion of nondegeneracy is equivalent to the property that $\mathcal{Q}$ and the affine set $\mathcal{A}$ intersect *transversally* at $\mathbf{x}$. This notion simply means that the tangent spaces at $\mathbf{x}$ to the two manifolds $\mathcal{A}$ and $\mathcal{Q}$ span $\Re^n$. This requirement for instance excludes the possibility that the affine set $\mathcal{A}$ is tangent to $\mathcal{Q}$. Now for $\mathcal{A}$, the tangent space is simply Ker $A = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ for all points in $\mathcal{A}$. Therefore,

**Definition 17** *Let $\mathcal{T}_\mathbf{x}$ be the tangent space at $\mathbf{x}$ to $\mathcal{Q}$. Then a primal-feasible point $\mathbf{x}$ is primal nondegenerate if*

$$\mathcal{T}_\mathbf{x} + \text{Ker}\,(A) = \Re^n; \tag{48}$$

*otherwise $\mathbf{x}$ is primal degenerate.*

The vector of free variables $\mathbf{y}$ in the dual problem in (2) can be eliminated by choosing any $(n-m)\times n$ sub-matrix $F$ so that the $n \times n$ matrix $G = (A^\top, F^\top)$ is nonsingular, and then pre-multiplying the equality constraint by $G^{-1} = (Y, Z)^\top$, where $Y$ and $Z$ have $m$ and $n-m$ columns respectively. This yields an equivalent system of equations

$$\begin{aligned} \mathbf{y} + Y^\top \mathbf{z} &= Y^\top \mathbf{c} \\ Z^\top \mathbf{z} &= Z^\top \mathbf{c} \end{aligned}$$

Since Span $(A^\top) = $ Ker $(Z^\top)$, where Span $(A^\top)$ is the linear space spanned by the rows of $A$, we obtain the following

**Definition 18** *A dual-feasible point $(\mathbf{y};\mathbf{z})$ is dual nondegenerate if*

$$\mathcal{T}_\mathbf{z} + \text{Span}\,(A^\top) = \Re^n; \tag{49}$$

*otherwise $(\mathbf{y};\mathbf{z})$ is dual degenerate.*

To algebraically characterize nondegeneracy, we need the following easily proved lemma.

**Lemma 19** *For given matrices $B$ and $C$, the matrix*

$$\begin{pmatrix} B & C \\ \mathbf{0}^\top & \mathbf{v}^\top \end{pmatrix}$$

*has full row rank for all nonzero $\mathbf{v}$ if and only if $B$ has full row rank.*

For the SOCP primal-dual pair (2), without loss of generality, we assume that all blocks $\mathbf{x_i} \in$ bd $\mathcal{Q}$ are grouped together in $\mathbf{x_B}$; all blocks $\mathbf{x_i} \in$ int $\mathcal{Q}$ are grouped together in $\mathbf{x_I}$; and all blocks $\mathbf{x_i} = \mathbf{0}$ are grouped together in $\mathbf{x_O}$. Hence the primal SOCP feasible point $\mathbf{x}$ may be partitioned into three parts:

$$\mathbf{x} = (\mathbf{x_B}; \mathbf{x_I}; \mathbf{x_O})$$

with $\mathbf{x_B} \in \Re^{n_B}$, $\mathbf{x_I} \in \Re^{n_I}$, and $\mathbf{x_O} \in \Re^{n_O}$; thus, $n_B + n_I + n_O = n$. We further assume that $\mathbf{x_B}$ has $p$ blocks:

$$\mathbf{x_B} = (\mathbf{x_1}; \cdots; \mathbf{x_p})$$

The coefficient matrix $A$ is also partitioned in the same manner; that is

$$A = \big(A_B, A_I, A_O\big) \text{ and } A_B = \big(A_1, \cdots, A_p\big).$$

Let $\mathcal{Q}_{n_i}$ be a single block second-order cone in $\Re^{n_i}$. Now observe that for $\mathbf{x} \in \operatorname{int} \mathcal{Q}_{n_i}$, the tangent space to $\mathcal{Q}_{n_i}$ is simply the whole $\Re^{n_i}$. For $\mathbf{x} = \mathbf{0}$, the tangent space is $\{\mathbf{0}\}$. Finally for $\mathbf{x} \in \operatorname{bd} \mathcal{Q}_{n_i}$, one can write $\mathbf{x} = \alpha\mathbf{c}'$ where $\mathbf{c}'$ along with $\mathbf{c} = R\mathbf{c}'$ form a Jordan frame. In that case the tangent space at $\mathbf{x}$ is the $n-1$ dimensional space $\big\{\mathbf{y} \mid \mathbf{c}^\top\mathbf{y} = 0\big\}$. In particular the orthogonal complement to the tangent space at $\mathbf{x}$ is the ray $\alpha\mathbf{c}$. For multiple block $\mathcal{Q}$, noting that tangent spaces are obtained by Cartesian product of tangent spaces for each block, we get the following characterization of primal nondegeneracy (PND):

$$\big(\mathcal{T}_{\mathbf{x_B}} \times \mathcal{T}_{\mathbf{x_I}} \times \mathcal{T}_{\mathbf{x_O}}\big) + \operatorname{Ker}\Big(\big(A_B, A_I, A_O\big)\Big) = \Re^n;$$

hence taking the orthogonal complement

$$\big((\alpha_1\mathbf{c_1}) \times \cdots \times (\alpha_p\mathbf{c_p}) \times \{\mathbf{0}\} \times \Re^{n_O}\big) \cap \operatorname{Span}\Big(\big(A_1, \cdots, A_p, A_I, A_O\big)^\top\Big) = \{\mathbf{0}\}$$

Therefore all matrices of the form

$$H_\mathcal{Q} = \begin{pmatrix} A_1 & \cdots & A_p & A_I & A_O \\ \alpha_1\mathbf{c_1}^\top & \cdots & \alpha_p\mathbf{c_p}^\top & \mathbf{0}^\top & \mathbf{v}^\top \end{pmatrix} \tag{50}$$

have linearly independent rows for all $\alpha_1, \cdots, \alpha_p$ and $\mathbf{v}$ not all zero. Now recall that for any $\mathbf{x_i} = \lambda_1\mathbf{c_i}' + \lambda_2\mathbf{c_i}$, the columns of the orthogonal matrix

$$Q_i = (\sqrt{2}\mathbf{c_i}, \hat{Q}_i, \sqrt{2}\mathbf{c_i}'),$$

where $\hat{Q}_i \in \Re^{n_i \times (n_i - 2)}$ is a matrix whose columns are orthogonal to $\mathbf{c_i}'$ and $\mathbf{c_i}$, are the eigenvectors of $\operatorname{Arw}(\mathbf{x_i})$.

**Theorem 20** *For each boundary block* $\mathbf{x_i} = \alpha_i\mathbf{c_i}'$, *let* $Q_i = (\sqrt{2}\mathbf{c_i}, \hat{Q}_i, \sqrt{2}\mathbf{c_i}') = (\sqrt{2}\mathbf{c_i}, \overline{Q_i})$ *be the matrix of eigenvectors of* $\operatorname{Arw}(\mathbf{x_i})$. *Then* $\mathbf{x} = \big(\mathbf{x_1}; \cdots; \mathbf{x_p}; \mathbf{x_I}; \mathbf{x_O}\big)$ *is primal nondegenerate if and only if the matrix*

$$\big(A_1\overline{Q_1}, \cdots, A_p\overline{Q_p}, A_I\big)$$

*has linearly independent rows. In particular,* $n_B + n_I - p \geq m$.

**Proof:** Set

$$G_\mathcal{Q} = Q_1 \oplus \cdots \oplus Q_p \oplus I \oplus I. \tag{51}$$

Since $Q_i^\top\mathbf{c_i} = \frac{\mathbf{e}}{\sqrt{2}}$, $i = 1, \ldots, p$, multiplying $H_\mathcal{Q}$ from the right by $G_\mathcal{Q}$ yields

$$H_\mathcal{Q}G_\mathcal{Q} = \begin{pmatrix} \mathbf{b_1} & A_1\overline{Q_1} & \cdots & \mathbf{b_p} & A_p\overline{Q_p} & A_I & A_O \\ \gamma_1 & \mathbf{0}^\top & \cdots & \gamma_p & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{v}^\top \end{pmatrix},$$

where $\mathbf{b_i} = \sqrt{2}A_i\mathbf{c_i}$ and $\gamma_i = \frac{\alpha_i}{\sqrt{2}}$. $H_\mathcal{Q}G_\mathcal{Q}$ has full row rank for all $\gamma_1, \cdots, \gamma_p$ and $\mathbf{v}$, not all zero, if and only if PND holds. Application of Lemma 19 completes the proof. ∎

Similarly let the feasible dual slack solution $\mathbf{z}$ be partitioned into three parts:

$$\mathbf{z} = (\mathbf{z_B}; \mathbf{z_O}; \mathbf{z_I})$$

with $\mathbf{z_B} \in \Re^{m_B}$, $\mathbf{z_O} \in \Re^{m_O}$, $\mathbf{z_I} \in \Re^{m_I}$. The block $\mathbf{z_B}$ is the concatenation of all those blocks of $\mathbf{z}$ that are on bd $\mathcal{Q}$ and it is partitioned into $q$ blocks

$$\mathbf{z_B} = (\mathbf{z_1}; \cdots; \mathbf{z_q}).$$

All blocks $\mathbf{z_i} = \mathbf{0}$ are concatenated as $\mathbf{z_O}$, and all blocks $\mathbf{z_i} \in \text{int } \mathcal{Q}$ are concatenated as $\mathbf{z_I}$. Thus, $m_B + m_O + m_I = n$. We also assume that $A$ is partitioned in the same way as $\mathbf{z}$; that is

$$A = \left(\tilde{A}_B, \tilde{A}_O, \tilde{A}_I\right) \text{ and } \tilde{A}_B = \left(\tilde{A}_1, \cdots, \tilde{A}_q\right).$$

($\tilde{A}$ is used to distinguish the partition induced by $\mathbf{z}$ from the one induced by $\mathbf{x}$.)

Dual nondegeneracy (DND) means

$$(\mathcal{T}_{\mathbf{z_B}} \times \mathcal{T}_{\mathbf{z_O}} \times \mathcal{T}_{\mathbf{z_I}}) + \text{Span}\left((\tilde{A}_B, \tilde{A}_O, \tilde{A}_I)^\top\right) = \Re^n.$$

For each boundary block let $\mathbf{z_i} = \beta_i \mathbf{d}_\mathbf{i}'$, where $\mathbf{d_i} = R\mathbf{d}_\mathbf{i}'$ and $\mathbf{d}_\mathbf{i}'$ form a Jordan frame. Then taking the orthogonal complement

$$\left((\beta_1 \mathbf{d_1}) \times \cdots \times (\beta_q \mathbf{d_q}) \times \Re^{m_I} \times \{\mathbf{0}\}\right) \cap \text{Ker}\left(\tilde{A}_1, \cdots, \tilde{A}_q, \tilde{A}_O, \tilde{A}_I\right) = \{\mathbf{0}\},$$

which means that if

$$\beta_1 \tilde{A}_1 \mathbf{d_1} + \cdots + \beta_q \tilde{A}_q \mathbf{d_q} + \tilde{A}_O \mathbf{v} = \mathbf{0},$$

then $\beta_i = 0$ for $i = 1, \ldots, q$, and $\mathbf{v} = \mathbf{0}$. Therefore we have

**Theorem 21** *The dual feasible solution* $(\mathbf{y}; \mathbf{z})$ *with* $\mathbf{z} = (\mathbf{z_1}; \cdots; \mathbf{z_q}; \mathbf{z_O}; \mathbf{z_I})$ *is dual nondegenerate if and only if the matrix*

$$\left(\tilde{A}_1 R\mathbf{z_1}, \cdots, \tilde{A}_q R\mathbf{z_q}, \tilde{A}_O\right)$$

*has linearly independent columns. In particular* $m \geq m_O + q$.

As in linear programming, at an optimal point primal nondegeneracy implies a unique dual solution (DUQ) and dual nondegeneracy implies primal uniqueness (PUQ).

**Theorem 22** *For the pair of primal-dual SOCP problems* (2)

*i. If a primal optimal solution is nondegenerate, then the optimal dual solution is unique.*

*ii. If a dual optimal solution is nondegenerate, then the optimal primal* *solution is unique.*

**Proof:** We shall prove the contrapositive of both statements.

i. Let $(\mathbf{y}';\mathbf{z}')$ and $(\mathbf{y}'';\mathbf{z}'')$ be two distinct optimal dual solutions and $\mathbf{x}$ be an optimal primal solution. Define $(\mathbf{y};\mathbf{z}) = (\mathbf{y}'-\mathbf{y}'';\mathbf{z}'-\mathbf{z}'')$; hence $A^\top\mathbf{y}+\mathbf{z}=\mathbf{0}$ and $\mathbf{y}$ and $\mathbf{z}$ are both nonzero. Suppose that $\mathbf{x}$ is partitioned as $\mathbf{x} = (\mathbf{x_1};\ldots;\mathbf{x_p};\mathbf{x_I};\mathbf{x_O})$, (with $\mathbf{x_i} \in \operatorname{bd}\mathcal{Q}$, $\mathbf{x_O}=\mathbf{0}$ and $\mathbf{x_I} \succ_{\mathcal{Q}} \mathbf{0}$,) and conformally,

$$\begin{aligned} A &= (A_1,\ldots,A_p,A_I,A_O) \\ \mathbf{z} &= (\mathbf{z_1};\ldots;\mathbf{z_p};\mathbf{z_I};\mathbf{z_O}), \end{aligned}$$

where from complementary slackness applied to $\mathbf{x_i}$ and $\mathbf{z}'_\mathbf{i}$ and $\mathbf{z}''_\mathbf{i}$, $\mathbf{z_i} = \alpha_i R\mathbf{x_i}$, for some $\alpha_i \in \Re$ for $i=1,\ldots,p$, $\mathbf{z_I}=\mathbf{0}$ and $\mathbf{z_O}\in\Re^{n_O}$. Since $A^\top\mathbf{y}+\mathbf{z}=\mathbf{0}$, the matrix

$$\begin{pmatrix} A_1 & \cdots & A_p & A_I & A_O \\ \mathbf{z_1}^\top & \cdots & \mathbf{z_p}^\top & \mathbf{0}^\top & \mathbf{z_O}^\top \end{pmatrix}$$

has linearly dependent rows. For each boundary block $\mathbf{z_i} = \hat{\alpha}_i\mathbf{c_i}$ for some $\hat{\alpha}_i \in \Re$ assuming that $\mathbf{x_i} = \tilde{\alpha}_i\mathbf{c}'_\mathbf{i}$; hence this matrix is of the same form as $H_\mathcal{Q}$ of (50), and since it does not have full row rank, $\mathbf{x}$ is degenerate.

ii. Similarly let $\mathbf{x}'$ and $\mathbf{x}''$ be two distinct optimal primal solutions and set $\mathbf{x} = \mathbf{x}'-\mathbf{x}'' \neq \mathbf{0}$. Let $(\mathbf{y};\mathbf{z})$ be an optimal dual solution. Partition $\mathbf{z} = (\mathbf{z_1};\ldots;\mathbf{z_q};\mathbf{z_O};\mathbf{z_I})$, (with $\mathbf{z_i}\in\operatorname{bd}\mathcal{Q}$, $\mathbf{z_O}=\mathbf{0}$, and $\mathbf{z_I}\succ_\mathcal{Q}\mathbf{0}$,) and let it induce the partitions

$$\begin{aligned} A &= (\tilde{A}_1,\ldots,\tilde{A}_q,\tilde{A_O},\tilde{A_I}) \\ \mathbf{x} &= (\mathbf{x_1};\ldots;\mathbf{x_q};\mathbf{x_O};\mathbf{x_I}), \end{aligned}$$

where by complementary slackness $\mathbf{x_i} = \beta_i R\mathbf{z_i}$ for $\beta_i\in\Re$, $\mathbf{x_I}=\mathbf{0}\in\Re^{m_O}$ and $\mathbf{x_O}\in\Re^{m_I}$. Since $A\mathbf{x}=\mathbf{0}$, we can write

$$\beta_1\tilde{A}_1R\mathbf{z_1}+\ldots+\beta_q\tilde{A}_qR\mathbf{z_q}+\tilde{A_O}\mathbf{x_O}=\mathbf{0}.$$

But this implies that the columns of matrix $(\tilde{A}_1R\mathbf{z_1},\ldots,\tilde{A}_qR\mathbf{z_q},\tilde{A_O})$ are linearly dependent. Therefore by Theorem 21 $(\mathbf{y},\mathbf{z})$ must be dual degenerate.

■

We now define strict complementarity for SOCP. First recall from linear programming that at the optimum, the pair $(\mathbf{x},\mathbf{z})$ satisfies strict complementarity if and only if $x_i+z_i>0$. This can be extended in a natural way to SOCP:

**Definition 23** *Let* $\mathbf{x}$ *and* $(\mathbf{y},\mathbf{z})$ *be optimal primal and dual solutions for the SOCP problem* (2). *Then* $(\mathbf{x},\mathbf{y},\mathbf{z})$ *satisfies strict complementarity (SC) if and only if* $\mathbf{z}+\mathbf{x}\in\operatorname{int}\mathcal{Q}$.

Note that the definition above is equivalent to $\mathbf{x_i} + \mathbf{z_i} \in \operatorname{int} \mathcal{Q}_{n_i}$ for each block $i$. With respect to its cone $\mathcal{Q}_{n_i}$ each block $\mathbf{x_i}$ may be in one of three states: Either it is in the interior of $\mathcal{Q}_{n_i}$ or it is in $\operatorname{bd} \mathcal{Q}_{n_i}$ or it equals $\mathbf{0}$. For optimal $\mathbf{x_i}$ and $\mathbf{z_i}$ any one of the following six states is possible:

| $\mathbf{x_i}$ | | $\mathbf{z_i}$ | SC |
|---|---|---|---|
| $\mathbf{x_i} \in \operatorname{int} \mathcal{Q}_{n_i}$ | $\Longrightarrow$ | $\mathbf{z_i} = \mathbf{0}$ | yes |
| $\mathbf{x_i} = \mathbf{0}$ | $\Longleftarrow$ | $\mathbf{z_i} \in \operatorname{int} \mathcal{Q}_{n_i}$ | yes |
| $\mathbf{x_i} \in \operatorname{bd} \mathcal{Q}_{n_i}$ | | $\mathbf{z_i} \in \operatorname{bd} \mathcal{Q}_{n_i}$ | yes |
| $\mathbf{x_i} \in \operatorname{bd} \mathcal{Q}_{n_i}$ | | $\mathbf{z_i} = \mathbf{0}$ | no |
| $\mathbf{x_i} = \mathbf{0}$ | | $\mathbf{z_i} \in \operatorname{bd} \mathcal{Q}_{n_i}$ | no |
| $\mathbf{x_i} = \mathbf{0}$ | | $\mathbf{z_i} = \mathbf{0}$ | no |

Notice that by the complementary slackness theorem, if $\mathbf{x_i}$ or $\mathbf{z_i}$ is in the interior of $\mathcal{Q}_{n_i}$, the other necessarily equals $\mathbf{0}$; that is

$$n_I \leq m_O \text{ and } m_I \leq n_O$$

**Corollary 24** $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *satisfies strict complementarity if for each block $i$, either both $\mathbf{x_i}$ and $\mathbf{z_i}$ are nonzero and in* $\operatorname{bd} \mathcal{Q}_{n_i}$*, or if one is zero, the other is in the interior of $\mathcal{Q}_{n_i}$.*

It can be easily verified that at the optimum $z_{i0} \det(\mathbf{x_i}) = x_{i0} \det(\mathbf{z_i}) = 0$. Thus strict complementarity holds if and only if for all blocks exactly one of $x_{i0}$ or $\det(\mathbf{z_i})$ is zero, and exactly one of $z_{i0}$ or $\det(\mathbf{x_i})$ is zero. Another characterization states that strict complementarity holds if and only if $m_B = n_B$, $m_I = n_O$, $m_O = n_I$, and $p = q$. In fact

**Corollary 25** *If $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is an optimal solution satisfying primal and dual nondegeneracy and strict complementarity, then,*

$$n_B + n_I - p \geq m \geq n_I + p.$$

This is in contrast with linear programming where PND, DND and SC imply that there is a unique optimal $\mathbf{x}$ and $\mathbf{z}$, which respectively have $m$ nonzero and $m$ zero components. Two SOCPs with identical block structure but with different data may have at their optima different numbers of blocks equal to zero, on the boundary or in the interior of their corresponding $\mathcal{Q}_{n_i}$, while both satisfying PND, DND and SC.

Also unlike linear programming, primal and dual nondegeneracy together do not imply strict complementarity. Consider the following QCQP example:

$$\begin{aligned} \min_{\mathbf{x} \in \Re^2} \quad & -x_1 \\ \text{s.t.} \quad & q_1(\mathbf{x}) \stackrel{\text{def}}{=} \|\mathbf{x}\|^2 \leq 1 \\ & q_2(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{x}^\top G \mathbf{x} \leq 4, \end{aligned}$$

where

$$G = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

It is easy to verify that the above QCQP has the unique solution $\mathbf{x} = (x_1; x_2) = (1; 0)$. Following the procedure described in §2, the above QCQP can be transformed into the following SOCP which is displayed along with its dual:

P: $\quad \min \quad -x_1$

$$\left(\begin{array}{cccccccc} 1 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 2 & 1 & \vdots & 0 & -1 & 0 \\ 0 & 0 & 2 & \vdots & 0 & 0 & -1 \\ 0 & 0 & 0 & \vdots & 1 & 0 & 0 \end{array}\right) \begin{pmatrix} \mathbf{x} \\ \mathbf{s} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$\mathbf{x} \succeq_{\mathcal{Q}} \mathbf{0}, \quad \mathbf{s} \succeq_{\mathcal{Q}} \mathbf{0}.$$

D: $\quad \max \quad y_1 + 2y_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \mathbf{y} + \mathbf{z_x} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

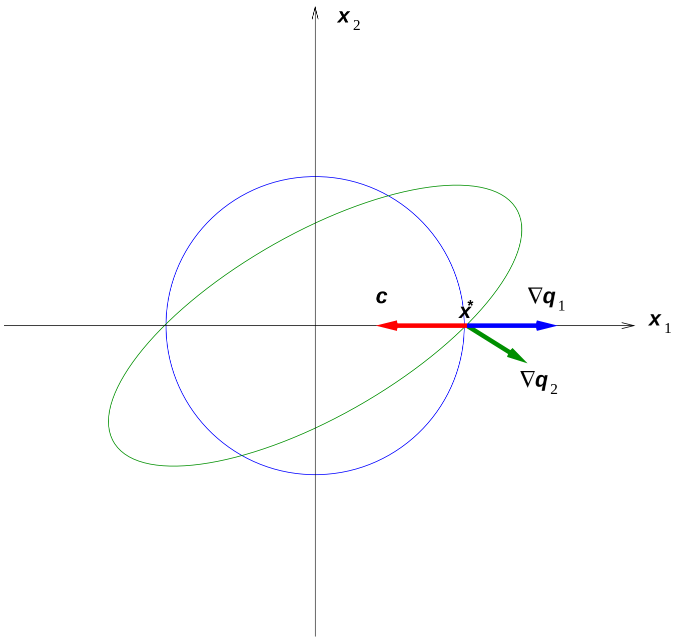
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \mathbf{y} + \mathbf{z_s} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{z_x} \succeq_{\mathcal{Q}} 0, \quad \mathbf{z_s} \succeq_{\mathcal{Q}} \mathbf{0}.$$

Here, $\mathbf{x} = (x_0; x_1; x_2)$ and $\mathbf{s} = (s_0; s_1; s_2)$. This pair of problems has a unique optimal solution pair

$$\mathbf{x}^* = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{s}^* = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \mathbf{y}^* = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{z}^*_{\mathbf{x}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{z}^*_{\mathbf{s}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

but, $(\mathbf{s}^*, \mathbf{z}^*_{\mathbf{s}})$ violates strict complementarity. This occurs even though the two constraints $q_1(\mathbf{x}) \leq 1$ and $q_2(\mathbf{x}) \leq 4$ intersect transversally at $\mathbf{x}^*$, and at this point the primal and dual nondegeneracy conditions hold.




Notice that $\mathbf{c} = \alpha\nabla q_1 + 0\nabla q_2$, which implies that the constraint $q_2(\mathbf{x}) \leq 4$ can be dropped from the problem without affecting its solution (i.e. this constraint is *weakly active*.) Such a situation cannot happen if all constraints are linear.

## 6.1 Nonsingularity of the Jacobian

If $\mathcal{Q}$ is the Cartesian product of $r$ second-order cones, primal and dual feasibility and complementarity give rise to the following system of equations:

$$\begin{array}{ll} A_i^\top \mathbf{y} + \mathbf{z_i} = \mathbf{c_i} & 1 \leq i \leq r \\ A_1\mathbf{x_1} + \cdots + A_n\mathbf{x_n} = \mathbf{b} & \\ \operatorname{Arw}(\mathbf{x_i})\operatorname{Arw}(\mathbf{z_i})\,\mathbf{e} = \mathbf{0} & 1 \leq i \leq r. \end{array} \tag{52}$$

In this section we show that when both primal and dual nondegeneracy and strict complementarity hold, then the Jacobian of the system (52) is nonsingular at the solution. In our study we need to deal with a particular type of block matrix:

**Definition 26** *Let*

$$J \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & B_1^\top & I & 0 \\ 0 & 0 & B_2^\top & 0 & I \\ B_1 & B_2 & 0 & 0 & 0 \\ V_1 & 0 & 0 & U_1 & 0 \\ 0 & V_2 & 0 & 0 & U_2 \end{pmatrix} \tag{53}$$

*where the first, second, third, fourth, and fifth block rows and columns have dimensions $m$, $n-m$, $m$, $m$, and $n-m$, respectively. We say $J$ is a* primal-dual block canonical *matrix (PDBC matrix for short) if*

1. $B_1 \in \Re^{m\times m}$ *is a nonsingular matrix,*
2. $V_2 \in \Re^{(n-m)\times(n-m)}$ *and* $U_1 \in \Re^{m\times m}$ *are symmetric positive definite,*
3. $V_1 \in \Re^{m\times m}$ *and* $U_2 \in \Re^{(n-m)\times(n-m)}$ *are symmetric positive semidefinite,*
4. $V_1$ *and* $U_1$ *commute, and likewise* $V_2$ *and* $U_2$ *commute.*

**Lemma 27** *Every primal-dual block canonical matrix is nonsingular.*

**Proof:** (This proof is essentially the same as the proof of Theorem 1 of [AHO98].) Interchange first and third rows and second and last columns. Then, since $B_1$ and $V_2$ are nonsingular, one can apply block Gaussian elimination to get

$$\begin{pmatrix} B_1 & 0 & 0 & 0 & B_2 \\ 0 & I & B_2^\top & 0 & 0 \\ 0 & 0 & B_1^\top & I & 0 \\ 0 & 0 & 0 & U_1 & -V_1 B_1^{-1} B_2 \\ 0 & 0 & 0 & 0 & C \end{pmatrix},$$

where

$$C = V_2 \left( I + V_2^{-1} U_2 B_2^\top B_1^{-T} U_1^{-1} V_1 B_1^{-1} B_2 \right).$$

Since $V_2$ is nonsingular we are only left to show that $I+V_2^{-1}U_2B_2^\top B_1^{-T}U_1^{-1}V_1B_1^{-1}B_2$ is nonsingular. But since $V_2$ and $U_2$ commute $V_2^{-1}U_2$ is symmetric positive semidefinite. Similarly, since $V_1$ and $U_1$ commute $B_2^\top B_1^{-T}U_1^{-1}V_1B_1^{-1}B_2$ is symmetric positive semidefinite. The product of two symmetric positive semidefinite matrices, though generally nonsymmetric, has real nonnegative eigenvalues and thus adding the identity gives a nonsingular matrix whose eigenvalues are all real and positive. ∎

Now the Jacobian of (52) is the matrix

$$J_Q = \left( \begin{array}{ccccc|c|ccccc}
0 & \cdots & 0 & 0 & 0 & A_1^\top & I & & & & \\
\vdots & \ddots & \vdots & 0 & 0 & \vdots & & \ddots & & & \\
0 & \cdots & 0 & 0 & 0 & A_p^\top & & & I & & \\
0 & \cdots & 0 & 0 & 0 & A_I^\top & & & & I & \\
0 & \cdots & 0 & 0 & 0 & A_O^\top & & & & & I \\
\hline
A_1 & \cdots & A_p & A_I & A_O & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
Z_1 & & & & & 0 & X_1 & & & & \\
 & \ddots & & & & \vdots & & \ddots & & & \\
 & & Z_p & & & 0 & & & X_p & & \\
 & & & Z_O & & 0 & & & & X_I & \\
 & & & & Z_I & 0 & & & & & X_O
\end{array} \right) \tag{54}$$

where $Z_i = \mathrm{Arw}(\mathbf{z_i})$, $X_i = \mathrm{Arw}(\mathbf{x_i})$, $Z_I = \mathrm{Arw}(\mathbf{z}_I)$, $Z_O = \mathrm{Arw}(\mathbf{z}_O)$, $X_I = \mathrm{Arw}(\mathbf{x}_I)$ and $X_O = \mathrm{Arw}(\mathbf{x}_O)$. Observe that $Z_O = 0$, $X_O = 0$, and $Z_I$ and $X_I$ are symmetric positive definite matrices. To show that $J_Q$ is nonsingular when primal and dual nondegeneracy and strict complementarity hold we transform it into PDBC form.

By complementary slackness optimal $\mathbf{x}$ and $\mathbf{z}$ operator commute; hence by Theorem 6 $\mathrm{Arw}(\mathbf{x})$ and $\mathrm{Arw}(\mathbf{z})$ commute and share a system of eigenvectors. In fact for optimal $(\mathbf{x_i}, \mathbf{z_i})$ the only way that neither $\mathbf{x_i}$ nor $\mathbf{z_i}$ is zero, is when both are on the boundary of $\mathcal{Q}_i$. In that case there is a Jordan frame $\mathbf{c}'_\mathbf{i}$ and $\mathbf{c_i} = R\mathbf{c}'_\mathbf{i}$, such that $\mathbf{x_i} = \alpha_i \mathbf{c}'_\mathbf{i}$ and $\mathbf{z_i} = \beta_i \mathbf{c_i}$, with $\alpha_i = x_{i0} + \|\overline{\mathbf{x}}_\mathbf{i}\| = 2x_{i0} > 0$ and $\beta_i = z_{i0} + \|\overline{\mathbf{z}}_\mathbf{i}\| = 2z_{i0} > 0$. Setting as before $Q_i = \left(\sqrt{2}\mathbf{c}'_\mathbf{i}, \hat{Q}_i, \sqrt{2}\mathbf{c_i}\right)$, the common orthogonal matrix of eigenvectors of $\mathrm{Arw}(\mathbf{x_i})$ and $\mathrm{Arw}(\mathbf{z_i})$ for each block $i$, we see that

$$Q_i^\top \mathrm{Arw}(\mathbf{x_i})\, Q_i = \begin{pmatrix} 2x_{i0} & \mathbf{0}^\top & 0 \\ \mathbf{0} & x_{i0} I & \mathbf{0} \\ 0 & \mathbf{0}^\top & 0 \end{pmatrix}$$

and

$$Q_i^\top \mathrm{Arw}(\mathbf{z_i})\, Q_i = \begin{pmatrix} 0 & \mathbf{0}^\top & 0 \\ \mathbf{0} & z_{i0} I & \mathbf{0} \\ 0 & \mathbf{0}^\top & 2z_{i0} \end{pmatrix}.$$

**Theorem 28** *For the SOCP problem, primal and dual nondegeneracy together with strict complementarity imply that the Jacobian matrix $J_\mathcal{Q}$ is nonsingular at the optimum.*

**Proof:** Define $P_\mathcal{Q}$ as the block diagonal matrix

$$P_\mathcal{Q} \stackrel{\text{def}}{=} (Q_1 \oplus \cdots \oplus Q_p \oplus I \oplus I) \oplus I \oplus (Q_1 \oplus \cdots \oplus Q_p \oplus I \oplus I) \tag{55}$$

with $Q_i$ corresponding to the eigenvectors of $\mathrm{Arw}(\mathbf{x_i})$ for boundary block $\mathbf{x_i}$. Consider the matrix $P_\mathcal{Q}^\top J_\mathcal{Q} P_\mathcal{Q}$. Primal and dual nondegeneracy and strict complementarity imply

i the matrix $A' \stackrel{\text{def}}{=} \left(A_1 \overline{Q_1}, \cdots, A_p \overline{Q_1}, A_I\right)$ has linearly independent rows, and $m \leq n_B + n_I - p$;

ii the matrix $A'' \stackrel{\text{def}}{=} (A_1 \mathbf{c_1}, \cdots, A_p \mathbf{c_p}, A_I)$, where $\mathbf{c_i}$ are as in Theorem 20, has linearly independent columns and thus $m \geq p + n_I$.

Therefore it is possible to take the $p + n_I$ columns of $A''$ together with some $m - p - n_I$ columns of $A'$ and build an $m \times m$ nonsingular matrix $B_1$. The remaining columns of $(A_1 Q_1, \cdots, A_p Q_p, A_I, A_O)$ form $B_2$. Once $B_1$ and $B_2$ are assigned $U_1$, and $U_2$ and $V_1$ and $V_2$ are forced. Notice that the blocks corresponding to $V_2$ all either arise from diagonal matrices corresponding to $Z_i$ with columns corresponding to their zero eigenvalues removed, or are blocks of $Z_I$;

in either case they are positive definite. Similarly blocks corresponding to $U_1$ arise from those columns of the $X_i$'s that have positive eigenvalues or from $X_I$, and thus they are positive definite. The remaining blocks of $X_i$ and $Z_i$ are assigned to $U_2$ and $V_1$. Thus $P_Q^\top J_Q P_Q$ is a PDBC matrix, which implies that $J_Q$ is nonsingular. ■

# 7 Primal-Dual Interior Point Methods

## 7.1 Background and connection to linear programming

We now turn our attention to the study of primal-dual interior point algorithms for the SOCP problem (2).

In linear programming there are generally two broad classes of interior point algorithms. One class consists of what can be roughly called primal-only or dual-only methods. It includes Karmarkar's original algorithm [Kar84], and many of the algorithms developed in the first few years after Karmarkar's paper appeared. The second class consists of primal-dual methods developed by [KMY89] and [MA89]. Roughly speaking, these algorithms apply Newton's method to the system of equations $A\mathbf{x} = \mathbf{b}, A^\top \mathbf{y} + \mathbf{z} = \mathbf{c}$, and the relaxed complementarity conditions $x_i z_i = \mu$. Extensive empirical study has shown that methods based on this primal-dual approach have favorable numerical properties over the primal-only or dual-only methods in general.

The first class of interior point methods mentioned can be generalized to SOCP rather effortlessly. In fact Nemirovski and Scheinberg have shown that Karmarkar's original algorithm extends essentially, word for word, to SOCP. The same can also be said for extensions of this class of algorithms to semidefinite programming. Alizadeh and Schmieta in [AS00] have shown that word for word extensions of the first class can be carried through to all symmetric cone optimization problems, of which semidefinite and second-order cones are special cases.

Extension of primal-dual methods from LP to SOCP and SDP, on the other hand, presents us with a challenge. First the natural generalization of $x_i z_i = \mu$ to SOCP is $\mathbf{x_i} \circ \mathbf{z_i} = \mu \mathbf{e}$. However, the proof presented in [MA89] cannot be extended word for word to SOCP since it relies in a crucial way on the fact that $\mathrm{Diag}(\mathbf{x})$ and $\mathrm{Diag}(\mathbf{z})$ always commute, as do all other analyses whether they are for path-following or potential-function reduction methods. The analog of $\mathrm{Diag}(\cdot)$ for SOCP is $\mathrm{Arw}(\cdot)$, but in general $\mathrm{Arw}(\mathbf{x})$ and $\mathrm{Arw}(\mathbf{z})$ do not commute. The situation is similar in SDP and in fact in all optimization problems over symmetric cones other than the nonnegative orthant. In order to alleviate the non-commutativity problem in semidefinite programming, researchers have proposed many different primal-dual methods. There is the Nesterov-Todd class developed and proved to have polynomial-time iteration complexity in [NT97, NT98]. (These methods were actually developed for optimization problems over symmetric cones.) There is the class of $XZ$, or $ZX$ methods which were developed by Helmberg et al. [HRVW96], Kojima et al. [KSH97] and Monteiro[Mon97]; the latter two papers gave polynomial-time iterations complexity proofs. Finally there is the $XZ + ZX$ method of [AHO98], which was subsequently shown to have polynomial time complexity by Monteiro [Mon98]. Then Zhang [Zha98] and Monteiro [Mon97] developed the Monteiro-Zhang family which includes all of the methods discussed above as special cases. Later on these classes were extended to all symmetric cone optimization problems. For instance Faybusovich [Fay97b, Fay97a, Fay98] extended the Nesterov-Todd and the $XZ$ and $ZX$ methods, and Schmieta and Alizadeh [SA01, SA99] extended the entire Monteiro-Zhang family to all symmetric cones.

In the following subsection we outline the development of primal-dual interior point algorithms for SOCP based on the algebraic properties given earlier. We focus exclusively on *path-following* methods.

## 7.2 Primal-dual path following methods

Note that for $\mathbf{x} \in \operatorname{int} \mathcal{Q}$ the function $-\ln\det(\mathbf{x})$ is a convex barrier function for $\mathcal{Q}$. To see this, note that by part of 6 Theorem 8, $\nabla^2_{\mathbf{x}}(-\ln\det(\mathbf{x})) = 2Q_{\mathbf{x}^{-1}}$. But, if $\lambda_{1,2} > 0$ are eigenvalues of $\mathbf{x}$, then $\lambda_{1,2}^{-1} > 0$ are eigenvalues of $\mathbf{x}^{-1}$. Therefore, by part 3 of Theorem 3 $Q_{\mathbf{x}^{-1}}$ has positive eigenvalues and hence is positive definite.

If we replace the second-order cone inequalities $\mathbf{x_i} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$ by $\mathbf{x_i} \succ_{\mathcal{Q}} \mathbf{0}$ and add the logarithmic barrier term $-\mu \sum_i \ln\det(\mathbf{x_i})$ to the objective function in the primal problem we get

$$
\begin{array}{lll}
(\mathbf{P}_\mu) & \min & \sum_{i=1}^r \mathbf{c_i}^\top \mathbf{x_i} - \mu \sum_{i=1}^r \ln\det(\mathbf{x_i}) \\
 & \text{s.t.} & \sum_{i=1}^r A_i \mathbf{x_i} = \mathbf{b} \\
 & & \mathbf{x_i} \succ_{\mathcal{Q}} \mathbf{0}, \text{ for } i = 1, \ldots, r.
\end{array} \tag{56}
$$

The Karush-Kuhn-Tucker(KKT) optimality conditions for (56) are:

$$
\sum_{i=1}^r A_i \mathbf{x_i} = \mathbf{b} \tag{57}
$$

$$
\mathbf{c_i} - A_i^\top \mathbf{y} - 2\mu \mathbf{x_i}^{-1} = \mathbf{0}, \quad \text{for } i = 1, \ldots, r \tag{58}
$$

$$
\mathbf{x_i} \succ_{\mathcal{Q}} \mathbf{0}, \quad \text{for } i = 1, \ldots, r. \tag{59}
$$

Set $\mathbf{z_i} \stackrel{\text{def}}{=} \mathbf{c_i} - A_i^\top \mathbf{y}$. Thus any solution of $\mathbf{P}_\mu$ satisfies:

$$
\begin{array}{l}
\sum_i A_i \mathbf{x_i} = \mathbf{b} \\
A_i^\top \mathbf{y} + \mathbf{z_i} = \mathbf{c_i} \quad \text{for } i = 1, \ldots, r \\
\mathbf{x_i} \circ \mathbf{z_i} = 2\mu \mathbf{e} \quad \text{for } i = 1, \ldots, r \\
\mathbf{x_i}, \mathbf{z_i} \succ_{\mathcal{Q}} \mathbf{0}, \text{ for } i = 1, \ldots, r.
\end{array} \tag{60}
$$

Similarly replacing $\mathbf{z_i} \succcurlyeq_{\mathcal{Q}} \mathbf{0}$ by $\mathbf{z_i} \succ_{\mathcal{Q}} \mathbf{0}$, in the dual and adding the logarithmic barrier $\mu \sum_i \ln\det(\mathbf{z_i})$ to the dual objective results in:

$$\begin{array}{lll} (\mathbf{D}_\nu) & \max & \mathbf{b}^\top \mathbf{y} + \mu \sum_{i=1}^r \ln\det(\mathbf{z_i}) \\ & \text{s.t.} & A_i^\top \mathbf{y} + \mathbf{z_i} = \mathbf{c_i}, \text{ for } i = 1, \dots, r, \\ & & \mathbf{z_i} \succ_{\mathcal{Q}} \mathbf{0}, \text{ for } i = 1, \dots, r. \end{array} \tag{61}$$

It is easily verified that the KKT conditions for (61) are (60), the same as for (56). For every $\mu > 0$ we can show that (60) has a unique solution. Thus we can define the notion of the *primal-dual central path* or simply the *central path*:

**Definition 29** *The trajectory of points* $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *satisfying* (60) *for* $\mu > 0$ *is the central path associated with the problem* (2).

Broadly speaking the strategy of path-following primal-dual interior point methods can be sketched as follows. We start with a point near (or on) the central path. First we apply Newton's method to the system (60) to get a direction $(\Delta\mathbf{x}, \Delta\mathbf{y}, \Delta\mathbf{z})$ that reduces the duality gap, and take a step in this direction making sure that the new point in still feasible and in the interior of $\mathcal{Q}$. Then we reduce $\mu$ by a constant factor and repeat the process. With judicious choices of initial point, step length and reduction schedule for $\mu$ we are able to show convergence in a polynomial number of iterations. We will not develop a complete analysis here; however, we will provide enough detail to give the basic ideas.

First let us explicitly derive the formulas for the direction resulting from application of Newton's method. If in (60) we replace every occurrence of $\mathbf{x_i}$, $\mathbf{y}$ and $\mathbf{z_i}$ with $\mathbf{x_i} + \Delta\mathbf{x_i}$, $\mathbf{y} + \Delta\mathbf{y}$, and $\mathbf{z_i} + \Delta\mathbf{z_i}$, expand the terms and eliminate all nonlinear terms in the $\Delta$'s we arrive at the following system of equations:

$$\sum_i A_i \Delta\mathbf{x_i} = \mathbf{b} - \sum_i A_i \mathbf{x_i}, \tag{62}$$

$$A_i^\top \Delta\mathbf{y} + \Delta\mathbf{z_i} = \mathbf{c_i} - A_i^\top \mathbf{y} - \mathbf{z_i}, \qquad \text{for } i = 1, \dots, r, \tag{63}$$

$$\mathbf{z_i} \circ \Delta\mathbf{x_i} + \mathbf{x_i} \circ \Delta\mathbf{z_i} = 2\mu\mathbf{e} - \mathbf{x_i} \circ \mathbf{z_i}, \qquad \text{for } i = 1, \dots, r. \tag{64}$$

We can write this system of equations in the block matrix form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ \operatorname{Arw}(\mathbf{z}) & 0 & \operatorname{Arw}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{y} \\ \Delta\mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{r}_p \\ \mathbf{r}_d \\ \mathbf{r}_c \end{pmatrix}, \tag{65}$$

where $\mathbf{x}, \mathbf{z}, \mathbf{c}, \Delta\mathbf{x}, \Delta\mathbf{z}$ are as in (1) and $\operatorname{Arw}(\cdot)$ is meant in the direct sum sense as described in §1. Also

$$\mathbf{r}_p \stackrel{\text{def}}{=} \mathbf{b} - A\mathbf{x}, \qquad \mathbf{r}_d \stackrel{\text{def}}{=} \mathbf{c} - A^\top \mathbf{y} - \mathbf{z}, \qquad \mathbf{r}_c \stackrel{\text{def}}{=} 2\mu\mathbf{e} - \mathbf{x} \circ \mathbf{z}.$$

Notice that $\mathbf{r}_p = \mathbf{0}$ if $\mathbf{x}$ is primal feasible, and $\mathbf{r}_d = \mathbf{0}$ if $(\mathbf{y}, \mathbf{z})$ is dual feasible.

Applying block Gaussian elimination to (65) yields:

$$\Delta\mathbf{y} = \left(A\operatorname{Arw}^{-1}(\mathbf{z}) \operatorname{Arw}(\mathbf{x}) A^\top\right)^{-1} \left(\mathbf{r}_p + A\operatorname{Arw}^{-1}(\mathbf{z}) \left(\operatorname{Arw}(\mathbf{x}) \mathbf{r}_d - \mathbf{r}_c\right)\right), \tag{66}$$

$$\Delta\mathbf{z} = \mathbf{r}_d - A^\top \Delta\mathbf{y} \tag{67}$$

$$\Delta\mathbf{x} = -\operatorname{Arw}^{-1}(\mathbf{z}) \left(\operatorname{Arw}(\mathbf{x}) \Delta\mathbf{z} - \mathbf{r}_c\right). \tag{68}$$

We will discuss efficient and numerically stable methods of calculating $(\Delta\mathbf{x}, \Delta\mathbf{y}, \Delta\mathbf{z})$ in §8.

It is not hard to show that the Newton direction (66)-(68) indeed reduces the duality gap. However, the main difficulty is ensuring that a sufficiently large step in that direction can be taken at *every iteration* in order to guarantee fast convergence. If for instance, in the first iteration we move nearly as far as possible in the Newton direction (that is, very close to the boundary of the feasible set), it may not be possible to move very far in subsequent iterations. To circumvent this problem in linear programming and subsequently in SDP and in symmetric cone programming, certain neighborhoods around the central path are defined. They are designed to accomplish two things: first, that a sufficiently large step length in the Newton direction can be taken without leaving the neighborhood, and second, the large step length guarantees a sufficiently large reduction in the duality gap. With these two properties in place it becomes easy to prove the polynomial-time iteration complexity of a method.

Let $\mathbf{w_i} = Q_{\mathbf{x_i}^{1/2}}\mathbf{z_i}$, and $\mathbf{w} = (\mathbf{w_1}; \ldots; \mathbf{w_r})$. Consider the following three *centrality measures*:

$$d_F(\mathbf{x},\mathbf{z}) \stackrel{\text{def}}{=} \|Q_{\mathbf{x}^{1/2}}\mathbf{z} - \mu\mathbf{e}\|_F = \sqrt{\sum_{i=1}^{r}\big(\lambda_1(\mathbf{w_i}) - \mu\big)^2 + \big(\lambda_2(\mathbf{w_i}) - \mu\big)^2} \quad (69)$$

$$d_2(\mathbf{x},\mathbf{z}) \stackrel{\text{def}}{=} \|Q_{\mathbf{x}^{1/2}}\mathbf{z} - \mu\mathbf{e}\|_2 = \max_{i=1,\ldots,r}\{|\lambda_1(\mathbf{w_i}) - \mu|, |\lambda_2(\mathbf{w_i}) - \mu|\} \quad (70)$$

$$d_{-\infty}(\mathbf{x},\mathbf{z}) \stackrel{\text{def}}{=} \mu - \min_{i=1,\ldots,r}\{\lambda_1(\mathbf{w_i}), \lambda_2(\mathbf{w_i})\}. \quad (71)$$

To justify these definitions, recall that the $d_\bullet(\mathbf{x},\mathbf{z})$ are functions of $x_j z_j$ in LP and $\lambda_i(XZ)$, i.e. eigenvalues of $XZ$ in SDP. But, $\lambda_i(XZ) = \lambda_i\big(X^{1/2}ZX^{1/2}\big)$. As we mentioned in §4, $Q_{\mathbf{x}}$ is the counterpart of the operator that sends $Y$ to $XYX$ in symmetric matrix algebra. Therefore, $Q_{\mathbf{x}^{1/2}}\mathbf{z}$ is the counterpart of $X^{1/2}ZX^{1/2}$. In fact, explicitly,

$$Q_{\mathbf{x}^{1/2}} = \begin{pmatrix} x_0 & \bar{\mathbf{x}}^T \\ \bar{\mathbf{x}} & \det^{1/2}(\mathbf{x})I + \frac{\bar{\mathbf{x}}\bar{\mathbf{x}}^T}{\det^{1/2}(\mathbf{x})+x_0} \end{pmatrix},$$

as was noted in [MT00]. Furthermore, by part 2 of Theorem 10 $Q_{\mathbf{x}^{1/2}}\mathbf{z}$ and $Q_{\mathbf{z}^{1/2}}\mathbf{x}$ have the same spectrum, which implies that the measures $d_\bullet(\mathbf{x},\mathbf{z})$ are symmetric with respect to $\mathbf{x}$ and $\mathbf{z}$.

Now we can define neighborhoods around the central path with respect to each of the centrality measures defined above. Let $\gamma \in (0,1)$.

$$\mathcal{N}_F(\gamma) \stackrel{\text{def}}{=} \{(\mathbf{x};\mathbf{y};\mathbf{z}) \mid A\mathbf{x} = \mathbf{b}, A^T\mathbf{y} + \mathbf{z} = \mathbf{c}, \quad \mathbf{x},\mathbf{z} \succ_{\mathcal{Q}} \mathbf{0}, d_F(\mathbf{x},\mathbf{z}) \le \gamma\mu\}, \quad (72)$$

$$\mathcal{N}_2(\gamma) \stackrel{\text{def}}{=} \{(\mathbf{x};\mathbf{y};\mathbf{z}) \mid A\mathbf{x} = \mathbf{b}, A^T\mathbf{y} + \mathbf{z} = \mathbf{c}, \quad \mathbf{x},\mathbf{z} \succ_{\mathcal{Q}} \mathbf{0}, d_2(\mathbf{x},\mathbf{z}) \le \gamma\mu\}, \quad (73)$$

$$\mathcal{N}_{-\infty}(\gamma) \stackrel{\text{def}}{=} \{(\mathbf{x};\mathbf{y};\mathbf{z}) \mid A\mathbf{x} = \mathbf{b}, A^T\mathbf{y} + \mathbf{z} = \mathbf{c}, \quad \mathbf{x},\mathbf{z} \succ_{\mathcal{Q}} \mathbf{0}, d_{-\infty}(\mathbf{x},\mathbf{z}) \le \gamma\mu\}. \quad (74)$$

Observe that since $d_F(\mathbf{x},\mathbf{z}) \geq d_2(\mathbf{x},\mathbf{z}) \geq d_{-\infty}(\mathbf{x},\mathbf{z})$, it follows that

$$\mathcal{N}_F(\mathbf{x},\mathbf{z}) \subseteq \mathcal{N}_2(\mathbf{x},\mathbf{z}) \subseteq \mathcal{N}_{-\infty}(\mathbf{x},\mathbf{z}).$$

Therefore, intuitively it may be more desirable to work with $\mathcal{N}_{-\infty}$ since it is larger and we have more room to move in the Newton direction at each iteration. However, it turns out that proving polynomial time convergence for this neighborhood is more difficult. In fact the best theoretical results are known for the smallest neighborhood $\mathcal{N}_F$.

We now discuss an effective way of scaling the primal-dual pair (2) to an *equivalent* pair of problems; this approach results in the so-called *Monteiro-Zhang* family of directions originally proposed by Monteiro [Mon97] and Zhang [Zha98] for SDP, and later generalized to all symmetric cones in [SA99].

Let $\mathbf{p} \succ_{\mathcal{Q}} \mathbf{0}$. With respect to $\mathbf{p}$, define

$$\begin{aligned}\widetilde{\mathbf{u}} &\stackrel{\text{def}}{=} Q_{\mathbf{p}}\mathbf{u} \text{ and}\\ \underset{\sim}{\mathbf{u}} &\stackrel{\text{def}}{=} Q_{\mathbf{p}^{-1}}\mathbf{u}.\end{aligned}$$

These definitions are valid for both the single and multiple block cases. Note that since $Q_{\mathbf{p}}Q_{\mathbf{p}^{-1}} = I$, as operators, $\widetilde{\cdot}$ and $\underset{\sim}{\cdot}$ are inverses of each other. Let us make the following change of variables: $\mathbf{x} \to \widetilde{\mathbf{x}}$. Recall from Theorem 9 that under this transformation $\mathcal{Q}$ remains invariant. With this change of variables the primal-dual pair (2) becomes

$$\begin{array}{llll}
\widetilde{P} & & \underset{\sim}{D} & \\
\min & \underset{\sim}{\mathbf{c}}_{\mathbf{1}}^\top \widetilde{\mathbf{x}}_{\mathbf{1}} + \cdots + \underset{\sim}{\mathbf{c}}_{\mathbf{r}}^\top \widetilde{\mathbf{x}}_{\mathbf{r}} & \max & \mathbf{b}^\top \mathbf{y} \\
\text{s.t.} & \underset{\sim}{A}_1 \widetilde{\mathbf{x}}_{\mathbf{1}} + \cdots + \underset{\sim}{A}_r \widetilde{\mathbf{x}}_{\mathbf{r}} = \mathbf{b} & \text{s.t.} & \underset{\sim}{A}_i^\top \mathbf{y} + \underset{\sim}{\mathbf{z}}_{\mathbf{i}} = \underset{\sim}{\mathbf{c}}_{\mathbf{i}}, \quad \text{for } i = 1,\dots,r \\
 & \widetilde{\mathbf{x}}_{\mathbf{i}} \succcurlyeq_{\mathcal{Q}} \mathbf{0}, \quad \text{for } i = 1,\dots,r & & \underset{\sim}{\mathbf{z}}_{\mathbf{i}} \succcurlyeq_{\mathcal{Q}} \mathbf{0}, \quad \text{for } i = 1,\dots,r
\end{array} \tag{75}$$

where

$$\begin{aligned}
&\mathbf{z} \to \underset{\sim}{\mathbf{z}},\\
&\mathbf{c} \to \underset{\sim}{\mathbf{c}},\\
&A_i \to \underset{\sim}{A}_i = A_i Q_{\mathbf{p}_{\mathbf{i}}^{-1}} \quad \text{ and thus } \quad A \to \underset{\sim}{A} = AQ_{\mathbf{p}^{-1}}.
\end{aligned}$$

**Lemma 30** *For a given nonsingular* $\mathbf{p}$ *we have:*

*i.* $\mathbf{x}^\top \mathbf{z} = \widetilde{\mathbf{x}}^\top \underset{\sim}{\mathbf{z}}$.

*ii. For each vector* $\mathbf{u}$, $A\mathbf{u} = \mathbf{0}$ *if and only if* $\underset{\sim}{A}\widetilde{\mathbf{u}} = \mathbf{0}$.

*iii.* $d_\bullet(\mathbf{x},\mathbf{z}) = d_\bullet(\widetilde{\mathbf{x}},\underset{\sim}{\mathbf{z}})$ *for the* $F$, $2$, *and* $-\infty$ *centrality measures.*

*iv. Under the scaling given, the central path and the neighborhoods* $\mathcal{N}_F$, $\mathcal{N}_2$ *and* $\mathcal{N}_{-\infty}$ *remain invariant.*

**Proof:** Parts i. and ii. follow from part 4 of Theorem 8. Part iv. is a consequence of part iii. We now prove part iii. Let $\widetilde{\mathbf{w}}_{\mathbf{i}} = Q_{\widetilde{\mathbf{x}}^{1/2}}\utilde{\mathbf{z}}$. $d_\bullet(\mathbf{x},\mathbf{z})$ depends only on the spectrum of $\mathbf{w}_{\mathbf{i}}$ and $d_\bullet(\widetilde{\mathbf{x}},\utilde{\mathbf{z}})$ depends only on the spectrum of $\widetilde{\mathbf{w}}_{\mathbf{i}}$. By part 1 of Theorem 10 the vectors $Q_{\mathbf{x}^{1/2}}\mathbf{z}$ and $Q_{\widetilde{\mathbf{x}}^{1/2}}\utilde{\mathbf{z}}$ have the same spectrum, proving part iii. ■

**Lemma 31** $(\widetilde{\Delta\mathbf{x}}, \Delta\mathbf{y}, \utilde{\Delta\mathbf{z}})$ *solves the system of equations*

$$\begin{aligned}
&\utilde{A}\widetilde{\Delta\mathbf{x}} = \mathbf{b} - \utilde{A}\widetilde{\mathbf{x}}\\
&\utilde{A}^\top\Delta\mathbf{y} + \utilde{\Delta\mathbf{z}} = \utilde{\mathbf{c}} - \utilde{A}^\top\mathbf{y} - \utilde{\mathbf{z}}\\
&\widetilde{\Delta\mathbf{x}}\circ\utilde{\mathbf{z}} + \widetilde{\mathbf{x}}\circ\utilde{\Delta\mathbf{z}} = 2\mu\mathbf{e} - \widetilde{\mathbf{x}}\circ\utilde{\mathbf{z}}
\end{aligned} \tag{76}$$

*if and only if* $(\Delta\mathbf{x},\Delta\mathbf{y},\Delta\mathbf{z})$ *solves*

$$\begin{aligned}
&A\Delta\mathbf{x} = \mathbf{b} - A\mathbf{x}\\
&A^\top\Delta\mathbf{y} + \Delta\mathbf{z} = \mathbf{c} - A^\top\mathbf{y} - \mathbf{z}\\
&(Q_{\mathbf{p}}\Delta\mathbf{x})\circ(Q_{\mathbf{p}^{-1}}\mathbf{z}) + (Q_{\mathbf{p}}\mathbf{x})\circ(Q_{\mathbf{p}^{-1}}\Delta\mathbf{z}) = 2\mu\mathbf{e} - (Q_{\mathbf{p}}\mathbf{x})\circ(Q_{\mathbf{p}^{-1}}\mathbf{z}).
\end{aligned} \tag{77}$$

Note that $(\widetilde{\Delta\mathbf{x}}, \Delta\mathbf{y}, \utilde{\Delta\mathbf{z}})$ is the result of applying Newton's method to the primal and dual feasibility and complementarity ($\widetilde{\mathbf{x}}\circ\utilde{\mathbf{z}} = \mu\mathbf{e}$) relations arising from the scaled problem (75). The corresponding $(\Delta\mathbf{x},\Delta\mathbf{y},\Delta\mathbf{z})$ is different from the one derived in (66),(67) and (68). Indeed the former depends on $\mathbf{p}$ and the latter results as a special case when $\mathbf{p}=\mathbf{e}$.

For each choice of $\mathbf{p}$ one gets a different direction. In particular, if we choose $\mathbf{p}=\mathbf{e}$, then the resulting direction is analogous to the so-called $XZ+ZX$ or AHO method in SDP. Of special interest are the class of $\mathbf{p}$'s for which $\widetilde{\mathbf{x}}$ and $\utilde{\mathbf{z}}$ operator commute.

**Definition 32** *The set of directions* $(\Delta\mathbf{x},\Delta\mathbf{y},\Delta\mathbf{z})$ *arising from those* $\mathbf{p}$ *where* $\widetilde{\mathbf{x}}$ *and* $\utilde{\mathbf{z}}$ *operator commute is called the commutative class of directions; a direction in this class is called a* commutative direction .

It is clear that when $\mathbf{p}=\mathbf{e}$ the resulting direction is not in general commutative. However, the following choices of $\mathbf{p}$ result in commutative directions:

$$\begin{aligned}
&\mathbf{p} = \mathbf{z}^{1/2}, \qquad \mathbf{p} = \mathbf{x}^{-1/2},\\
&\mathbf{p} = \left[Q_{\mathbf{x}^{1/2}}\left(Q_{\mathbf{x}^{1/2}}\mathbf{z}\right)^{-1/2}\right]^{-1/2} = \left[Q_{\mathbf{z}^{-1/2}}\left(Q_{\mathbf{z}^{1/2}}\mathbf{x}\right)^{1/2}\right]^{-1/2}.
\end{aligned}$$

The first two are analogs of $XZ$ and $ZX$ directions (also referred to as the HRVW/KSH/M directions) in SDP and the third one is the Nesterov and Todd (NT) direction. Note that by parts 3, 4 and 7 of Theorem 8, for $\mathbf{p}=\mathbf{z}^{1/2}$

$$\utilde{\mathbf{z}} = Q_{\mathbf{p}^{-1}}\mathbf{z} = Q_{\mathbf{z}^{-1/2}}\left(\mathbf{z}^{1/2}\right)^2 = \left(Q_{\mathbf{z}^{-1/2}}Q_{\mathbf{z}^{1/2}}\right)\mathbf{e} = \mathbf{e},$$

and for $\mathbf{p} = \mathbf{x}^{-1/2}$

$$\widetilde{\mathbf{x}} = Q_{\mathbf{p}}\mathbf{x} = Q_{\mathbf{x}^{-1/2}}\big(\mathbf{x}^{1/2}\big)^2 = \big(Q_{\mathbf{x}^{-1/2}}Q_{\mathbf{x}^{1/2}}\big)\mathbf{e} = \mathbf{e};$$

while for the Nesterov-Todd direction $\mathbf{p}$,

$$\begin{aligned}Q_{\mathbf{p}}^2\mathbf{x} = Q_{\mathbf{p}^2}\mathbf{x} = \big(Q^{-1}_{Q_{\mathbf{x}^{1/2}}(Q_{\mathbf{x}^{1/2}}\mathbf{z})^{-1/2}}\big)\mathbf{x} = \big(Q_{Q_{\mathbf{x}^{-1/2}}(Q_{\mathbf{x}^{1/2}}\mathbf{z})^{1/2}}\big)\mathbf{x}\\ =\big(Q_{\mathbf{x}^{-1/2}}Q_{(Q_{\mathbf{x}^{1/2}}\mathbf{z})^{1/2}}Q_{\mathbf{x}^{-1/2}}\big)\mathbf{x} = \big(Q_{\mathbf{x}^{-1/2}}Q_{(Q_{\mathbf{x}^{1/2}}\mathbf{z})^{1/2}}\big)\mathbf{e} = \big(Q_{\mathbf{x}^{-1/2}}Q_{\mathbf{x}^{1/2}}\big)\mathbf{z} = \mathbf{z};\end{aligned}$$

hence $\widetilde{\mathbf{x}} = Q_{\mathbf{p}}\mathbf{x} = Q_{\mathbf{p}^{-1}}\mathbf{z} = \underset{\sim}{\mathbf{z}}$. It follows that in each of these cases $\widetilde{\mathbf{x}}$ and $\underset{\sim}{\mathbf{z}}$ operator commute.

We will now sketch the strategy of primal-dual path-following interior point methods that results in a polynomial-time iteration count for SOCP problems. Suppose that $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a feasible point in the appropriate neighborhood and that $(\Delta\mathbf{x}, \Delta\mathbf{y}, \Delta\mathbf{z})$ is the direction obtained by solving the system (77) with $\mu = \frac{\operatorname{tr}(\mathbf{x}\circ\mathbf{z})}{2r}\sigma$ for some constant $\sigma \in (0,1)$, which, in turn, leads to the new feasible point $(\mathbf{x}+\Delta\mathbf{x}, \mathbf{y}+\Delta\mathbf{y}, \mathbf{z}+\Delta\mathbf{z})$. Then

$$\begin{aligned}\operatorname{tr}\big((\mathbf{x}+\Delta\mathbf{x})\circ(\mathbf{z}+\Delta\mathbf{z})\big) &= \operatorname{tr}(\mathbf{x}\circ\mathbf{z}) + \operatorname{tr}\big(\Delta\mathbf{x}\circ\mathbf{z} + \mathbf{x}\circ\Delta\mathbf{z}\big) + \operatorname{tr}(\Delta\mathbf{x}\circ\Delta\mathbf{z})\\ &= \operatorname{tr}(\mathbf{x}\circ\mathbf{z}) + \operatorname{tr}\left(\sigma\frac{\operatorname{tr}(\mathbf{x}\circ\mathbf{z})}{2r}\mathbf{e} - \mathbf{x}\circ\mathbf{z}\right) + \operatorname{tr}(\Delta\mathbf{x}\circ\Delta\mathbf{z})\\ &= \sigma\operatorname{tr}(\mathbf{x}\circ\mathbf{z}) + 2\Delta\mathbf{x}^\top\Delta\mathbf{z}\end{aligned}$$

Since we assume that $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is feasible, $\mathbf{r}_p = \mathbf{0}$ and $\mathbf{r}_d = \mathbf{0}$, which implies $\Delta\mathbf{x}^\top\Delta\mathbf{z} = 0$. Thus at each iteration the duality gap is reduced by a factor of $\sigma$. The harder part is to choose $\sigma$ and $\gamma$ in such a way that the new point remains in the corresponding neighborhood $\mathcal{N}_\bullet(\gamma)$, thus ensuring feasibility throughout all iterations.

To summarize what is known about the complexity of path-following SOCP algorithms let us define the notion of *iteration complexity*. We say that an interior point algorithm has $\mathcal{O}(f)$ iteration complexity if every $f$ iterations reduce the size of the duality gap $\mathbf{x}^\top\mathbf{z}$ by at least a constant factor, say 1/2. Here is a summary of results for the iteration complexity of primal-dual path-following interior point methods for all cone-LPs over all symmetric cones and therefore for the second-order cone:

1. For the neighborhood $\mathcal{N}_F(\gamma)$ it can be shown that for all $\mathbf{p}$ the new point is also in $\mathcal{N}_F(\gamma)$ if we choose $\sigma = \left(1 - \frac{\delta}{\sqrt{r}}\right)$, for some $\gamma, \delta \in (0,1)$. In this case the iteration complexity of the algorithm is $\mathcal{O}(\sqrt{r})$ ([Mon98] for SDP, [MT00] for SOCP, and [SA01, SA99, Sch99] for optimization over representable symmetric cones).

2. For the commutative directions and neighborhoods $\mathcal{N}_2(\gamma)$ and $\mathcal{N}_{-\infty}(\gamma)$ one can show that the iteration complexity is $\mathcal{O}(\kappa r)$, again for $\sigma = \left(1 - \frac{\delta}{\sqrt{r}}\right)$, and suitable $\gamma, \delta \in (0,1)$. Here $\kappa$ is the least upper bound on the

condition number of the matrix $G^{1/2}$ in $\mathcal{N}_\bullet(\gamma)$, where $G = \operatorname{Arw}^{-1}(\underset{\sim}{\mathbf{z}})\operatorname{Arw}(\widetilde{\mathbf{x}})$. For the Nesterov-Todd direction $G = I$ the identity matrix, and so the iteration complexity is $\mathcal{O}(r)$ in the $\mathcal{N}_2$ and the $\mathcal{N}_{-\infty}$ neighborhoods, respectively. When $\mathbf{p} = \mathbf{x}^{-1/2}$ or $\mathbf{p} = \mathbf{z}^{1/2}$ one can show that $\kappa = \mathcal{O}(1)$ for $\mathcal{N}_2(\gamma)$ and $\kappa = \mathcal{O}(\sqrt{r})$ for $\mathcal{N}_{-\infty}(\gamma)$. Therefore, the iteration complexity is $\mathcal{O}(r)$ and $\mathcal{O}(r^{1.5})$, respectively for $\mathcal{N}_2(\gamma)$ and $\mathcal{N}_{-\infty}(\gamma)$ ([MZ98], [Tsu99], [SA99]). The complexity of non-commutative directions, in particular for the choice of $\mathbf{p} = \mathbf{e}$, remains unresolved with respect to the $\mathcal{N}_2(\gamma)$ and the $\mathcal{N}_{-\infty}(\gamma)$ neighborhoods.

# 8 Efficient and Numerically Stable Algorithms

We now consider implementation of interior point algorithms for solving SOCPs in an efficient and numerically stable manner. The most computationally intensive task at each iteration of an interior point method for solving an SOCP involves the solution of the system of equations (77) for computing the Newton search direction. After eliminating the $\Delta z$ variables from (77), one obtains a system of equations of the form

$$\begin{pmatrix} -F^{-1} & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix}, \tag{78}$$

where $F = Q_{\mathbf{p}^{-1}}\Big(\operatorname{Arw}^{-1}(\underset{\sim}{\mathbf{z}})\operatorname{Arw}(\widetilde{\mathbf{x}})\Big)Q_{\mathbf{p}^{-1}}$. In LP, the matrix $F$ is diagonal. In SOCP, $F$ is a diagonal matrix plus a generally nonsymmetric low rank matrix. To establish this fact first we state and prove the following lemma. For now let us assume that we are working with a single block SOCP.

**Lemma 33** *Let $\mathbf{x} \in \Re^n$ and $\mathbf{z} \in \Re^n$ be such that $\mathbf{z}$ is nonsingular. Then*

$$G(\mathbf{x},\mathbf{z}) = \operatorname{Arw}^{-1}(\mathbf{z})\operatorname{Arw}(\mathbf{x}) = \beta I - \gamma R\mathbf{x}\mathbf{e}^\top + \gamma R\mathbf{z}\mathbf{u}^\top \tag{79}$$

*where $\beta = \frac{x_0}{z_0}$, $\gamma = \frac{1}{z_0}$, $\mathbf{u} = \mathbf{x} \circ \mathbf{z}^{-1}$ and $\mathbf{e}$ is the identity vector.*

**Proof:** Noting that $\mathbf{z}^{-1} = \frac{1}{\det(\mathbf{z})}R\mathbf{z}$, this lemma can be proved by inspection. ∎

We are now ready for our main theorem for multiple block SOCPs.

**Theorem 34** *Let $\mathbf{x} \in \Re^n$ and $\mathbf{z} \in \Re^n$ be such that $\mathbf{z}$ is nonsingular. For all nonsingular $\mathbf{p}$ we have*

$$F = Q_{\mathbf{p}^{-1}}\Big(\operatorname{Arw}^{-1}(\underset{\sim}{\mathbf{z}})\operatorname{Arw}(\widetilde{\mathbf{x}})\Big)Q_{\mathbf{p}^{-1}} = D + T = D' + T'$$

*where $D$ is the direct sum of multiples of the identity matrix and $T$ is a nonsymmetric matrix in general, whose rank is at most $3r$; and $D'$ is the direct sum of multiples of $R$ and $T'$ is a nonsymmetric matrix in general of rank at most $2r$.*

**Proof:** First we prove the theorem for the single block case where $r = 1$. Multiplying $G(\tilde{\mathbf{x}}, \underset{\sim}{\mathbf{z}})$ in Lemma 33 from the left and the right by $Q_{\mathbf{p}^{-1}}$, setting $\mathbf{q} = \mathbf{p}^{-1}$, $\theta = \beta \det(\mathbf{q})^2$, and noting from parts 4, 5 and 8 of Theorem 8 that $Q_{\mathbf{q}}^2 = Q_{\mathbf{q}^2} = -\det(\mathbf{q}^2)R + 2\mathbf{q}^2\left(\mathbf{q}^2\right)^\top = \det(\mathbf{q})^2\left(I - 2\mathbf{e}\mathbf{e}^\top\right) + 2\mathbf{q}^2\left(\mathbf{q}^2\right)^\top$ and $\mathbf{e}^\top Q_{\mathbf{q}} = \left(\mathbf{q}^2\right)^\top$, we obtain

$$
\begin{aligned}
F &= Q_{\mathbf{q}}\left(\operatorname{Arw}^{-1}\left(\underset{\sim}{\mathbf{z}}\right)\operatorname{Arw}\left(\tilde{\mathbf{x}}\right)\right)Q_{\mathbf{q}} \\
&= \beta Q_{\mathbf{q}^2} - \gamma\left(Q_{\mathbf{q}}R\mathbf{x}\right)\left(\mathbf{q}^2\right)^\top + \gamma Q_{\mathbf{q}}R\mathbf{z}\mathbf{u}^\top Q_{\mathbf{q}} \\
&= \theta I - 2\theta\mathbf{e}\mathbf{e}^\top + \left(2\beta\mathbf{q}^2 - \gamma Q_{\mathbf{q}}R\mathbf{x}\right)\left(\mathbf{q}^2\right)^\top + \gamma Q_{\mathbf{q}}R\mathbf{z}\mathbf{u}^\top Q_{\mathbf{q}} \qquad (80) \\
&= -\theta R + \left(2\beta\mathbf{q}^2 - \gamma Q_{\mathbf{q}}R\mathbf{x}\right)\left(\mathbf{q}^2\right)^\top + \gamma Q_{\mathbf{q}}R\mathbf{z}\mathbf{u}^\top Q_{\mathbf{q}}. \qquad (81)
\end{aligned}
$$

The identity (80) determines $D$ and $T$, and (81) determines $D'$ and $T'$. Therefore, the theorem for the single block case is proved. For multiple blocks note that $F = F_1 \oplus \cdots \oplus F_r$ and hence, it is the direct sum of multiples of the identity matrix plus a matrix of rank at most $3r$, and the direct sum of multiples of $R$ plus a matrix of rank at most $2r$. ■

We note the following special cases:

**$\mathbf{p} = \mathbf{e}$:**

In this case $Q_{\mathbf{p}} = I$ and by Lemma 33 $T$ is nonsymmetric and has rank $2r$. Therefore, $F$ is the sum of a symmetric positive definite matrix $D$ which is the direct sum of multiples of the identity matrix and a nonsymmetric matrix of rank $2r$.

**The NT case:**

In the Nesterov-Todd method, recall that $\mathbf{p}$ is chosen to make $\underset{\sim}{\mathbf{z}} = \tilde{\mathbf{x}}$. Therefore, $G = \operatorname{Arw}^{-1}\left(\underset{\sim}{\mathbf{z}}\right)\operatorname{Arw}\left(\tilde{\mathbf{x}}\right) = I$ and $F = Q_{\mathbf{p}^{-2}} = Q_{\mathbf{q}^2}$ which is symmetric positive definite. Moreover, $T$ is symmetric and has rank $2r$. On the other hand $F$ can also be written as a direct sum of multiples of $R$, a diagonal but indefinite matrix, and a symmetric matrix $T'$ of rank $r$.

**The commutative class:**

When $\mathbf{p}$ is chosen to make $\underset{\sim}{\mathbf{z}}$ and $\tilde{\mathbf{x}}$ operator commute, the rank of $T$ is generally $3r$ (with the noted exception of the NT method), but $T$ is symmetric. Also $F$ is symmetric and positive semidefinite, and can be written as a direct sum of multiples of $R$, a diagonal but indefinite matrix, and a symmetric matrix of rank $2r$. This is true for instance for the choices of $\mathbf{p} = \mathbf{z}^{1/2}$ and $\mathbf{p} = \mathbf{x}^{-1/2}$.

In the remainder of this section we consider the commutative class of directions and in particular the NT direction. One can solve (78) directly by computing the Bunch-Parlett-Kaufmann factorization of the symmetric indefinite coefficient matrix in (78) (see [GL96] for details). However, the more standard approach involves eliminating the $\Delta\mathbf{x}$ variables using the first block of equations in (78) and then solving the so-called *normal equations*

$$(AFA^\top)\Delta\mathbf{y} = \mathbf{r}_2 + AF\mathbf{r}_1 \tag{82}$$

by computing the Cholesky factorization of the positive definite symmetric matrix $AFA^\top$. It is easy to see that $AFA^\top = \sum_{i=1}^r A_i F_i A_i^\top$, where in general, each $F_i = Q_{\mathbf{p_i}^{-2}}$ is totally dense.

We are interested in exploiting sparsity in the original data matrix $A$, when the dimension of at least one of the blocks is relatively large and there are only a small number of such large blocks. In this case $AFA^\top$ is dense or has a dense square sub-matrix of relatively large size even if $A$ is sparse; hence computing the factorization of $AFA^\top$ directly can be costly. Observe, however, that $F_i = Q_{\mathbf{p_i}^{-2}}$ can be expressed as

$$F_i = \det(\mathbf{q_i})^2 I + 2\mathbf{q_i}^2 \left(\mathbf{q_i}^2\right)^\top - 2\det(\mathbf{q_i})^2 \mathbf{e}\mathbf{e}^\top \tag{83}$$

where $\mathbf{q_i} = \mathbf{p_i}^{-1}$. Therefore, if we partition the index set of the blocks into those that are large, $\mathcal{L}$, and small, $\mathcal{S}$, and for $i \in \mathcal{L}$, $A_{i0}$ denotes the first column of $A_i$, we have that

$$AFA^\top = \sum_{i\in\mathcal{S}} A_i F_i A_i^\top + \sum_{i\in\mathcal{L}} \det(\mathbf{q_i})^2 A_i A_i^\top + \sum_{i\in\mathcal{L}} \mathbf{v_i}\mathbf{v_i}^\top - \sum_{i\in\mathcal{L}} \mathbf{w_i}\mathbf{w_i}^\top, \tag{84}$$

where

$$\mathbf{v_i} = \sqrt{2} A_i \left(\mathbf{q_i}^2\right), \quad \mathbf{w_i} = \sqrt{2}\det(\mathbf{q_i}) A_{i0}. \tag{85}$$

All terms in (84) are sparse except for the rank-one matrices $\mathbf{v_i}\mathbf{v_i}^\top$, assuming that the matrices $A_i A_i^\top$ are sparse. However, we can only compute the (sparse) Cholesky factorization $LDL^\top$ of the matrix equal to the first two sums in (84), which we shall denote by $M$, since including the term $\left(-\sum_{i\in\mathcal{L}} \mathbf{w_i}\mathbf{w_i}^\top\right)$ may result in an indefinite matrix. One could use the so-called Schur complement approach which is based on the Sherman-Morrison-Woodbury update formula (e.g., see [And96, CMS90, Wri97]) to handle the low rank matrix $VV^\top - WW^\top \stackrel{\text{def}}{=} \sum_{i\in\mathcal{L}}(\mathbf{v_i}\mathbf{v_i}^\top - \mathbf{w_i}\mathbf{w_i}^\top)$. Specifically, since $AFA^\top = LDL^\top + VV^\top - WW^\top$, the solution to (82) can be computed by solving

$$LE = (V, W)$$

and

$$L\mathbf{h} = \mathbf{r_2} + AF\mathbf{r_1}$$

for $E$ and $\mathbf{h}$; then forming

$$G = J + E^\top D^{-1} E,$$

where $J \stackrel{\text{def}}{=} I \oplus -I$ and $(V, W)J = (V, -W)$, and solving

$$G\mathbf{g} = E^\top D^{-1}\mathbf{h}$$

and

$$L^\top \Delta\mathbf{y} = D^{-1}(\mathbf{h} - E\mathbf{g}).$$

Unfortunately, as shown in [GS01b], this approach which has been used in interior point methods for LP to handle dense columns in the constraint matrix $A$, is not numerically stable if $LDL^\top$ is singular or nearly so, as is often the case when the optimal solution is approached.

This approach is even more numerically unstable in the SOCP case than it is in the LP case. It can be shown that as the duality gap $\mu \downarrow 0$, all the eigenvalues of $F_i$ equal $\mathcal{O}(\mu)$ or $\mathcal{O}(1/\mu)$, or one equals $\mathcal{O}(\mu)$ and one equals $\mathcal{O}(1/\mu)$ with the rest equal to $\mathcal{O}(1)$ depending on how the primal-dual pair $\mathbf{x_i}$ and $\mathbf{z_i}$ approach the boundaries of their respective cones, see [GS01a] and [AS97]. Hence, as in LP the condition number of $AFA^\top$ is $\mathcal{O}(1/\mu^2)$, i.e., $AFA^\top$ will be extremely ill-conditioned, as $\mu \downarrow 0$. However, the presence of intermediate-sized eigenvalues in the SOCP case results in additional loss of accuracy. Experiments on real world and randomly generated problems have verified this and the unsuitability of using the Schur complement approach.

## 8.1 A product-form Cholesky factorization approach

A numerically stable approach for updating the sparse factorization $LDL^\top$ is provided by the product-form Cholesky factorization (PFCF) approach first suggested by Goldfarb and Scheinberg [GS01b] for implementing interior point methods for LP. Like the Schur complement approach, the PFCF method first computes the Cholesky factorization $LDL^\top$ of the sparse matrix $PMP^\top$, where $M = AFA^\top - (VV^\top - WW^\top)$, $L$ is a unit lower triangular matrix, $D = \mathrm{Diag}(d_1, \ldots, d_m)$ is a positive diagonal matrix and $P$ is a permutation matrix chosen to reduce the amount of fill-in in the Cholesky factorization. Determining the "best" $P$ is NP-hard; but heuristics such as the minimum degree, minimum local fill-in and nested dissection orderings yield acceptable results on most sparse problems with a modest amount of effort (e.g., see [GL81]). If any of the matrices $A_i, i \in \mathcal{S}$ have dense columns, the blocks corresponding to them are treated as belonging to the set $\mathcal{L}$ and the outer products of these dense columns scaled by $\det(\mathbf{q_i^2})$ are included in $VV^\top$ rather than in $M$. Then the symmetric rank-one term corresponding to a dense column $\mathbf{v} = \mathbf{v_i}$ is added to $M = LDL^\top$, and the PFCF method computes $\tilde{L}$ and $\tilde{D}$, where

$$LDL^\top + \mathbf{v}\mathbf{v}^\top = L(D + \mathbf{h}\mathbf{h}^\top)L^\top = L\tilde{L}\tilde{D}\tilde{L}^\top L^\top, \tag{86}$$

$\mathbf{h} \in \Re^m$ is the solution to $L\mathbf{h} = \mathbf{v}$,

$$\tilde{L} = \tilde{L}(\mathbf{h}, \boldsymbol{\beta}) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & & & & & \\ h_2\beta_1 & 1 & & & & \\ h_3\beta_1 & h_3\beta_2 & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ h_{m-1}\beta_1 & h_{m-1}\beta_2 & h_{m-2}\beta_3 & \cdots & 1 & \\ h_m\beta_1 & h_m\beta_2 & h_m\beta_3 & \cdots & h_m\beta_{m-1} & 1 \end{pmatrix}, \tag{87}$$

and the diagonal elements $\tilde{d}_j$ of the $m \times m$ diagonal matrix $\tilde{D}$ and the components of $\boldsymbol{\beta} \in \Re^m$ are given by the following recurrence first proposed for quasi-Newton updates [FP74, GMS75]:

$$\begin{aligned} & t_0 := 1 \\ & \text{for } j = 1, 2, \ldots, m \\ & \qquad t_j := t_{j-1} + h_j^2/d_j, \\ & \qquad \tilde{d}_j := d_j t_j / t_{j-1}, \\ & \qquad \beta_j := h_j/(d_j t_j). \end{aligned} \tag{88}$$

The procedure that is actually used is more complicated than this since $M$ may be only positive semidefinite. Note that the Cholesky factorization of a positive semidefinite matrix can be obtained in a numerically stable manner [Hig96]. By repeating this procedure $k$ times, once for each dense column of $V$, we obtain the product-form factorization,

$$L\tilde{L}_1\tilde{L}_2 \ldots, \tilde{L}_k\tilde{D}^k\tilde{L}_k^\top \ldots \tilde{L}_2^\top\tilde{L}_1^\top L^\top \tag{89}$$

of $M + VV^\top$. The method for subtracting a rank-one term $\mathbf{w}\mathbf{w}^\top$ (which should be done after the corresponding $\mathbf{v}\mathbf{v}^\top$ term has been added) is similar; i.e.,

$$LDL^\top - \mathbf{w}\mathbf{w}^\top = \tilde{L}(D - \mathbf{h}\mathbf{h}^\top)L^\top = L\tilde{L}\tilde{D}\tilde{L}^\top L^\top, \tag{90}$$

where $L\mathbf{h} = \mathbf{w}$, $\tilde{L} \stackrel{\text{def}}{=} \tilde{L}(\mathbf{h}, \boldsymbol{\beta})$ is a special unit lower triangular matrix of the form (87), but the recurrence used to compute the diagonal elements of $\tilde{D}$ and the components of $\boldsymbol{\beta}$ is different from (87) (see [FP74]). This yields a product-form factorization of the form (89) where $k = 2|\mathcal{L}|$.

Since each matrix $\tilde{L}_\nu = \tilde{L}_\nu(\mathbf{h}^\nu, \beta^\nu), \nu = 1, \ldots, k$ in (89) is a special unit lower triangular matrix of the form (87), we need only store the pair of vectors $\mathbf{h}^\nu$ and $\boldsymbol{\beta}^\nu$ for it. Moreover, systems of the form $\tilde{L}_\nu\mathbf{s} = \mathbf{r}$ and $\tilde{L}_\nu^\top\mathbf{s} = \mathbf{r}$ can be solved using only $2(m-2)$ multiplications and $2(m-1)$ additions.

In [GS01a] it is proved that under assumptions that are almost always satisfied in practice, the PFCF procedure is numerically stable. This PFCF approach for SOCPs has been implemented in SeDuMi [Stu98], an efficient interior point code that handles a mixture of LP, SOCP and SDP constraints. It has enabled SeDuMi to solve large conic problems in a tiny fraction of time and memory that was previously needed as well as solve several formerly intractable problems (J. F. Sturm, private communication.)

Unfortunately, the PFCF method is not applicable to interior point methods that use directions such as the so-called $XZ + ZX$ direction [AHO98] for which $F$ is not symmetric. (This is the case $\mathbf{p} = \mathbf{e}$.) However, these matrices still have structure; e.g., in the $XZ + ZX$ case, $F = F_1 \oplus \ldots \oplus F_r$, each $F_i = G(\mathbf{x_i}, \mathbf{z_i})$ in Lemma 33; that is, each $F_i$ is the sum of $\frac{x_{i0}}{z_{i0}} I$ and two nonsymmetric outer products. Let us assume that there is at least one, but not too many, large blocks. If, in addition, there are many small blocks, a method proposed in [GS01c] first computes a sparse $\hat{L}\hat{U}$ factorization (ignoring permutations for simplicity) of $\sum_{\mathcal{S}} A_i F_i A_i^\top + \sum_{\mathcal{L}} (x_{i0}/z_{0i}) A_i A_i^\top$, while if there are very few small blocks it first computes a sparse Cholesky factorization $\hat{L}\hat{D}\hat{L}^\top$ of $\sum_{i=1}^N (x_{i0}/z_{i0}) A_i A_i^\top$. For each rank-one nonsymmetric matrix $\mathbf{v}\mathbf{w}^\top$ this method uses a combination of updating techniques originally proposed in [GMS75, Gol76] in another context. Given the current factorization $\hat{L}\hat{D}\hat{U}$, it computes

$$\hat{L}\hat{D}\hat{U} + \mathbf{v}\mathbf{w}^\top = \hat{L}(\hat{D} + \mathbf{h}\mathbf{s}^\top)\hat{U} = \hat{L}\tilde{L}\tilde{D}\tilde{V}\hat{U}, \tag{91}$$

where $\hat{L}\mathbf{h} = \mathbf{v}$. $\hat{U}^\top \mathbf{s} = \mathbf{w}$ and $\tilde{L}\tilde{D}\tilde{V}$ is closely related to an $LQ$ factorization of $\hat{D} + \mathbf{h}\mathbf{s}^\top$ that avoids the computation of square roots. This so-called $LDV$ factorization must be defined carefully if it is to be numerically stable. In particular, rather than requiring $\tilde{D}^{1/2}\tilde{V}$ to equal an orthogonal matrix $Q$, we require that $\tilde{D}^{1/2}\tilde{V}\hat{D}^{1/2} = Q$, or equivalently, that $\tilde{V}^\top \tilde{D}\tilde{V} = \hat{D}^{-1}$.

This approach is attractive because $\tilde{L}$ and $\tilde{V}$ are special matrices: $L = L(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{h}, \mathbf{s})$ is a unit lower triangular matrix whose $(i,j)$-th element $L_{i,j} = h_i \alpha_j + s_i \beta_j$, for $i > j$ and $\tilde{V}$ is the product of a special lower Hessenberg matrix $H(\boldsymbol{\tau}, t, \boldsymbol{\rho})$ whose $(i,j)$-th element $H_{i,j} = t_i \tau_j$, for $i \geq j$, and whose $(i, i+1)$-th element $H_{i,i+1} = -\rho_i$ for $i < n$, with the transpose of another such matrix $H(\mathbf{1}, \mathbf{w}, \boldsymbol{\sigma})$. The vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\tau}, \mathbf{t}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ and the diagonal elements of $\tilde{D}$ can all be computed using fairly simple recurrence formulas and only these vectors and $\mathbf{h}$ and $\mathbf{s}$ need to be stored.

# 9 Current and Future Research Directions

An area that was not covered in this paper is the application of SOCP to combinatorial optimization problems. Successful application of linear programming, and more recently semidefinite programming, to hard combinatorial optimization problems is well-known. However, while there are a few applications that utilize SOCP directly (notably the Fermat-Weber plant location problem mentioned in §2), there is little work on relaxing integer programs to SOCPs. In this regard we note the work of Kim and Kojima [KK01] and of Iyengar and Çezik [IÇ01]. One particular area for further development is the possible adoption of Gomory–Chvatal type cuts to SOCP inequalities yielding new valid SOCP inequalities.

An interesting question is whether there is a simplex-like algorithm for SOCP. One advantage that simplex algorithms have over interior point methods in LP is that once a problem has been solved, and subsequently changed in a small way (say by adding an extra constraint), the dual simplex method can be used to find the optimal solution starting from the original solution. Typically a small number of dual simplex iterations suffice, and each iteration takes $\mathcal{O}\left(m^2\right)$ arithmetic operations for dense problems and less for sparse problems. In interior point methods each iteration typically costs $\mathcal{O}\left(m^3\right)$ for dense problems; hence even one iteration is expensive. No satisfactory technique is known in interior point algorithms for using knowledge of the optimum solution to a closely related problem that requires only $\mathcal{O}\left(m^2\right)$ arithmetic operations. Presently no simplex-like algorithm for SOCP is known, nor have the active set methods of nonlinear programming been investigated for this class of problems.

### Acknowledgment

We thank the referees for their extremely thorough reviews, and for pointing out numerous typographical and algebraic errors in earlier versions of this paper.

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