Integer Programming and Arrovian Social Welfare Functions

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Abstract

We formulate the problem of deciding which preference domains admit a non-dictatorial Arrovian Social Welfare Function as one of verifying the feasibility of an integer linear program. Many of the known results about the presence or absence of Arrovian Social Welfare Functions, and properties of majority rules, can be derived in a simple and unified way from this integer program. We provide a complete polyhedral description of Arrovian Social Welfare Functions on single-peaked domains. Finally, we extend the method to study Arrovian Social Choice functions.

Keywords: Social Welfare Function, Impossibility Theorem, Single-Peaked Domain, Linear Programming

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1 Introduction

The Old Testament likens the generations of men to the leaves of a tree. It is a simile that applies as aptly to the literature inspired by Arrow's impossibility theorem [2]. Much of it is devoted to classifying those preference domains that admit or exclude the existence of a non-dictatorial Arrovian Social Welfare Function (ASWF).[1] We add another leaf to that tree. Here we formulate the problem of deciding whether a preference domain admits a non-dictatorial ASWF as an integer program. This formulation allows us to derive in a systematic way many of the known results about Arrovian domains.

The integer program derived here is inspired by a characterization of Arrovian domains due to Kalai and Muller [7]. We reformulate it as an integer program. So as to describe the result we introduce some notation.

Let A denote the set of alternatives (at least three). Let Σ denote the set of all transitive, antisymmetric and total binary relations on A. An element of Σ is a preference ordering. Notice that this set up excludes indifference. The set of admissible preference orderings for members of a society of n-agents will be a subset of Σ and denoted Ω. Let Ω^n be the set of all n-tuples of preferences from Ω. An element of Ω^n will typically be denoted as $\mathbf{P} = (\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n})$, where $\mathbf{p_i}$ is interpreted as the preference ordering of agent i. In the language of Le Breton and Weymark [14] we assume the **common preference domain** framework.

An n-person Social Welfare Function is a function $f : \Omega^n \to \Sigma$. Thus for any $\mathbf{P} \in \Omega^n$, $f(\mathbf{P})$ is an ordering of the alternatives. We write $xf(\mathbf{P})y$ if x is ranked above y under $f(\mathbf{P})$. An n-person **Arrovian Social Welfare Function** (ASWF) on Ω is a function $f : \Omega^n \to \Sigma$ that satisfies the following two conditions:

1. **Unanimity:** If for $\mathbf{P} \in \Omega^n$ and some $x, y \in A$ we have $x\mathbf{p_i}y$ for all i then $xf(\mathbf{P})y$.

2. **Independence of Irrelevant Alternatives:** For any $x, y \in A$ suppose $\exists \mathbf{P}, \mathbf{Q} \in \Omega^n$ such that $x\mathbf{p_i}y$ if an only if $x\mathbf{q_i}y$ for $i = 1, \ldots, n$. Then $xf(\mathbf{P})y$ if an only if $xf(\mathbf{Q})y$.

An ordered pair $x, y \in A$ is called **trivial** if $x\mathbf{p}y$ for all $\mathbf{p} \in \Omega$. In view of unanimity, any ASWF must have $xf(\mathbf{P})y$ for all $\mathbf{P} \in \Omega^n$ whenever x, y is a trivial pair. If Ω consists only of trivial pairs then distinguishing between dictatorial and non-dictatorial ASWF becomes nonsensical, so we assume that Ω contains at least one non-trivial pair. An ASWF is **dictatorial** if there is an i such that $f(\mathbf{P}) = \mathbf{p_i}$ for all $\mathbf{P} \in \Omega^n$. Call Ω **Arrovian** if it admits a non-dictatorial ASWF.

Kalai and Muller [7] derive a set of necessary and sufficient conditions on Ω for the existence of a non-dictatorial 2-person ASWF. They also prove that Ω admits an n-person non-dictatorial ASWF if and only if it admits a 2-person non-dictatorial ASWF. Thus, to decide if Ω is Arrovian it suffices to determine if a 2-person non-dictatorial ASWF exists.

The main contributions of this paper are summarized below.

[1] An ASWF is a social welfare function that satisfies the axioms of the Impossibility theorem.

- Our first result is an integer linear programming formulation of the problem of finding a n-person ASWF. For each Ω we construct a set of linear inequalities with the property that every feasible 0-1 solution corresponds to a n-person ASWF. For certain classes of Ω (e.g., Ω is single-peaked), the polytope defined by the set of linear inequalities is **integral**: the vertices of the polytope correspond to ASWF's and every ASWF corresponds to a vertex of the polytope.

- We use this formulation to derive a result of Kalai and Muller [7]: Ω admits an n-person non-dictatorial ASWF if and only if it admits a 2-person non-dictatorial ASWF. This allows us to derive a succinct linear integer program to decide if Ω is Arrovian. For the case when $|A| = 3, 4$ we derive the convex hull of integer solutions.

- The characterization of ASWF via integer program allows us to derive several structural results on majority rules in a simple manner. The same approach can also be used to study Arrovian Social Choice Function.

- The inequalities in the formulation can be divided into two groups: easy and hard. The easy ones have the property that they define a polyhedron whose extreme points are 0-1. Hence if the easy constraints were the only ones that mattered, the feasibility of the integer program would coincide with feasibility of its linear relaxation. The second set of constraints, the hard ones, introduce fractional extreme points to the underlying linear relaxation. This makes a characterization of the feasible 0-1 solutions of the set of inequalities difficult. Nevertheless, for many different subsets Ω of Σ we are able to show that the hard constraints are redundant. Under this condition one can establish that Ω is Arrovian if and only if the corresponding set of linear inequalities admits 0-1 solutions different from the all zero's and all one's solution. The extreme points of the inequality set characterize all ASWF's on this domain. The easy constraints are capable of a graph theoretic interpretation which allows one to formulate a theorem of the following kind: If a certain directed graph, depending on Ω, is strongly connected, then Ω is non-arrovian. Arrow's theorem, amongst others, is a consequence of this result.

2 The Integer Program

Denote the set of all ordered pairs of alternatives by A^2. Let E denote the set of all agents, and S^c denote $E \setminus S$ for all $S \subseteq E$.

To construct an n-person ASWF we exploit the independence of irrelevant alternatives condition. This allows us to specify an ASWF in terms of which ordered pair of alternatives a particular subset, S, of agents is decisive over.

Definition 1 *For a particular ASWF, a subset S of agents is* **weakly decisive for x over y** *if whenever all agents in S rank x above y and all agents in S^c rank y over x, the ASWF ranks x over y.*

Since this is the only notion of decisiveness used in the paper, we omit the qualifier 'weak' in what follows.

2.1 n-person ASWF's

For each non-trivial element $(x, y) \in A^2$, we define a 0-1 variable as follows:

$$d_S(x,y) = \begin{cases} 1 & \text{if the subset } S \text{ of agents is decisive for } x \text{ over } y, \\ 0 & \text{otherwise.} \end{cases}$$

If $(x, y) \in A^2$ is a trivial pair then by default we set $d_S(x, y) = 1$ for all $S \neq \emptyset$; also, for notational purposes we assume that $d_\emptyset(x, y) = 0$ for all $(x, y) \in A^2$.

For an ASWF f, we can determine whether $d_S(x, y) = 1$ or 0 by observing the output $f(\mathbf{P})$ of a suitably chosen $\mathbf{P} \in \Omega^n$ in which agents in S rank x over y, and agents in S^c rank y over x. In the rest of this section, we study the properties that these functions should satisfy.

Unanimity: To ensure unanimity,

$$d_E(x, y) = 1 \text{ for all } x, y. \tag{1}$$

Independence of Irrelevant Alternatives: Consider a pair of alternatives $(x, y) \in A^2$, a $\mathbf{P} \in \Omega^n$, and a subset S of agents such that all agents in S prefer x to y, and all agents in S^c prefer y to x. Suppose $xf(\mathbf{P})y$. Let $\mathbf{Q}$ be any other profile such that all agents in S rank x over y and all agents in S^c rank y over x. By the independence of irrelevant alternatives condition $xf(\mathbf{Q})y$. Hence the set S is decisive for x over y. However, had $yf(\mathbf{P})x$ a similar argument would imply that S^c is decisive for y over x. Hence for all S and $(x, y) \in A^2$, we must have

$$d_S(x, y) + d_{S^c}(y, x) = 1. \tag{2}$$

Transitivity: To motivate the next class of constraints, it is useful to consider majority rule. If the number n of agents is odd, majority rule can be described using the following variables:

$$d_S(x,y) = \begin{cases} 1 & \text{if } |S| > n/2, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that this solution satisfies both (1) and (2). However, if Ω admits a *Condorcet* triple (e.g., $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3} \in \Omega$ with $x\mathbf{p_1}y\mathbf{p_1}z$, $y\mathbf{p_2}z\mathbf{p_2}x$, and $z\mathbf{p_3}x\mathbf{p_3}y$), then such a rule does not always produce an *ordering* of the alternatives for each preference profile. Our next constraint is designed to rule out this and similar possibilities.

Let A, B, C, U, V, and W be (possibly empty) *disjoint* sets of agents whose union includes all agents. For each such partition of the agents, and any triple x, y, z,

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) \leq 2, \tag{3}$$

where the sets satisfy the following conditions (hereafter referred to as condition (*)):

$$A \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, x\mathbf{p}z\mathbf{p}y,$$
$$B \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, y\mathbf{p}x\mathbf{p}z,$$
$$C \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, z\mathbf{p}y\mathbf{p}x,$$
$$U \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, x\mathbf{p}y\mathbf{p}z,$$
$$V \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, z\mathbf{p}x\mathbf{p}y,$$
$$W \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega, y\mathbf{p}z\mathbf{p}x.$$

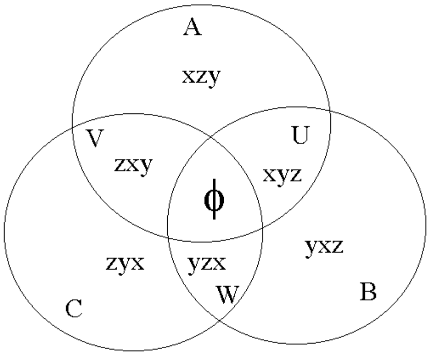

Figure 1: The sets and the associated orderings

It is easy to show that every ASWF satisfies (3). Suppose the contrary, and there is an ASWF f such that the corresponding d variables violate constraint (3). Then there exists (possibly empty) disjoint sets A, B, C, U, V and W, and triple x, y, z, such that

$$d_{A \cup U \cup V}(x,y) = d_{B \cup U \cup W}(y,z) = d_{C \cup V \cup W}(z,x) = 1,$$

with the sets A, B, C, U, V, and W satisfying condition (*). Note that in such a case, each of the three sets $A \cup U \cup V$, $B \cup U \cup W$, and $C \cup V \cup W$ must be non-empty.

We now construct a profile $\mathbf{P}$ as follows: agents in A rank x over z over y; similarly, agents in B rank y over x over z, agents in C rank z over y over x, agents in U rank x over y over z, agents in V rank z over x over y, and finally agents in W rank y over z over x. The agents in $A \cup U \cup V$, which is guaranteed to be non-empty, rank x over y and are decisive for x over y. Hence $xf(\mathbf{P})y$. The agents in $B \cup U \cup W$, also guaranteed to be non-empty, rank y over z, and are decisive for y over z; hence $yf(\mathbf{P})z$. By considering the agents in $C \cup V \cup W$, we conclude, using an identical argument, that $zf(\mathbf{P})x$. This contradicts the assertion that f is an ASWF.

A consequence of (3) combined with (2) that will be useful is that

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) \geq 1.$$

To see this, interchange the role of z and x in (3). Then the roles of A and V (resp. B and W, C and U) can be interchanged to obtain the new inequality:

$$d_{A\cup C\cup V}(z,y)+d_{B\cup C\cup W}(y,x)+d_{A\cup B\cup U}(x,z)\le 2.$$

Using (2), we obtain

$$d_{B\cup U\cup W}(y,z)+d_{A\cup U\cup V}(x,y)+d_{C\cup V\cup W}(z,x)\ge 1.$$

Subsequently we prove that constraints (1-3) are both necessary and sufficient for the characterization of n-person ASWF's. Before that, it is useful to develop a better understanding of constraints (3), and their relationship to the constraints identified in [7], called *decisiveness implications*, described below.

Suppose there are $\mathbf{p},\mathbf{q}\in\Omega$ and three alternatives x,y and z such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Kalai and Muller [7] showed that

$$d_S(x,y)=1\Rightarrow d_S(x,z)=1,$$

and

$$d_S(z,x)=1\Rightarrow d_S(y,x)=1.$$

These conditions can be formulated as the following two inequalities:

$$d_S(x,y)\le d_S(x,z), \tag{4}$$

$$d_S(z,x)\le d_S(y,x). \tag{5}$$

The first condition follows from using the profile $\mathbf{P}$ in which agents in S rank x over y over z and agents in S^c rank y over z over x. If S is decisive for x over y, then $xf(\mathbf{P})y$. By unanimity, $yf(\mathbf{P})z$. By transitivity, $xf(\mathbf{P})z$. Hence S is also decisive for x over z. The second condition follows from a similar argument.

Claim 1 *Constraints (4, 5) are special cases of constraints (3).*

Proof. Let

$$U\leftarrow S, W\leftarrow S^c$$

in constraint (3), with the other sets being empty. Note that U and W can be assumed to be non-empty by condition (*). Constraint (3) reduces to

$$d_U(x,y)+d_{U\cup W}(y,z)+d_W(z,x)\le 2.$$

Since $U\cup W=E$, the above reduces to

$$0\le d_S(x,y)+d_{S^c}(z,x)\le 1,$$

which implies $d_S(x,y) \leq d_S(x,z)$ by (2). By interchanging the roles of S and S^c, we obtain the inequality $d_{S^c}(x,y) \leq d_{S^c}(x,z)$, which is equivalent to $d_S(z,x) \leq d_S(y,x)$. ■

Suppose we know only that there is a $\mathbf{p} \in \Omega$ with $x\mathbf{p}y\mathbf{p}z$. In this instance, transitivity requires:

$$d_S(x,y) = 1 \textbf{ and } d_S(y,z) = 1 \Rightarrow d_S(x,z) = 1,$$

and

$$d_S(z,x) = 1 \Rightarrow \textbf{ at least one of } \; d_S(y,x) = 1 \textbf{ or } \; d_S(z,y) = 1.$$

These can be formulated as the following two inequalities:

$$d_S(x,y) + d_S(y,z) \;\leq\; 1 + d_S(x,z), \tag{6}$$

$$d_S(z,y) + d_S(y,x) \;\geq\; d_S(z,x). \tag{7}$$

Similarly, we have:

Claim 2 *Constraints (6, 7) are special cases of constraints (3).*

Proof. Suppose $\exists\ \mathbf{q} \in \Omega$ with $x\mathbf{q}y\mathbf{q}z$. If there exists $\mathbf{p} \in \Omega$ with $y\mathbf{p}z\mathbf{p}x$ or $z\mathbf{p}x\mathbf{p}y$, then constraints (6, 7) are implied by constraints (4, 5), which in turn are special cases of constraint (3). So we may assume that there does not exist $\mathbf{p} \in \Omega$ with $y\mathbf{p}z\mathbf{p}x$ or $z\mathbf{p}x\mathbf{p}y$. If there does not exist $\mathbf{p} \in \Omega$ with $z\mathbf{p}y\mathbf{p}x$, then x, z is a trivial pair, and constraints (6, 7) are redundant. So we may assume that such a $\mathbf{p}$ exists, hence C can be chosen to be non-empty. Let

$$U \leftarrow S, C \leftarrow S^c$$

in constraint (3), with the other sets being empty. Constraint (3) reduces to

$$d_U(x,y) + d_U(y,z) + d_C(z,x) \leq 2,$$

which is just

$$d_S(x,y) + d_S(y,z) + d_{S^c}(z,x) \leq 2.$$

Thus constraint (6) follows as a special case of constraint (3). By reversing the roles of S and S^c again, we can show similarly that constraint (7) follows as a special case of constraint (3). ■

In fact, for $n = 2$, it can be shown that constraints (1-3) reduce to constraints (1, 2, 4-7).

Theorem 1 *Every feasible integer solution to (1)-(3) corresponds to an ASWF and vice-versa.*

Proof. Given an ASWF, we have shown that the corresponding d vector satisfies (1)-(3). Now pick any feasible solution to (1)-(3) and call it d. To prove that d gives rise to an ASWF, we show that for every profile of preferences from Ω, d generates an ordering of the alternatives. Unanimity and Independence of Irrelevant Alternatives follow automatically from the way the d_S variables are used to construct the ordering.

Suppose d does not produce an ordering of the alternatives. Then, for some profile $\mathbf{P} \in \Omega^n$, there are three alternatives x, y and z such that d ranks x above y, y above z and z above x. For this to happen there must be three non-empty sets H, I, and J such that

$$d_H(x,y) = 1,\ d_I(y,z) = 1,\ d_J(z,x) = 1,$$

and for the profile $\mathbf{P}$, agent i ranks x over y (resp. y over z, z over x) if and only if i is in H (resp. I, J). Note that $H \cup I \cup J$ is the set of all agents, and $H \cap I \cap J = \emptyset$. Let

$$A \leftarrow H \setminus (I \cup J), B \leftarrow I \setminus (H \cup J), C \leftarrow J \setminus (H \cup I),$$

$$U \leftarrow H \cap I, V \leftarrow H \cap J, W \leftarrow I \cap J.$$

Now A (resp. B, C, U, V, W) can only be non-empty if there exists $\mathbf{p}$ in Ω with $x\mathbf{p}z\mathbf{p}y$ (resp. $y\mathbf{p}x\mathbf{p}z$, $z\mathbf{p}y\mathbf{p}x$, $x\mathbf{p}y\mathbf{p}z$, $z\mathbf{p}x\mathbf{p}y$, $y\mathbf{p}z\mathbf{p}x$).

In this case constraint (3) is violated since

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) = d_H(x,y) + d_I(y,z) + d_J(z,x) = 3.$$

■

A dictatorial ASWF, with agent j as dictator, corresponds to a solution of the system where $d_S(x,y) = 1$ for all $(x,y) \in A^2$ whenever $S \ni j$. It is easy to see that such an ASWF satisfies (1)-(3). To illustrate how (1)-(3) can be used to verify if a rule is an ASWF, we consider the born loser rule. For each j, we can define the *born loser* rule with respect to j (denoted by B_j) in the following way:

- $d_E^{B_j}(x,y) = 1$ for every $x,y \in A^2$.
- $d_\emptyset^{B_j}(x,y) = 1$ for every $x,y \in A^2$.
- For every non-trivial pair (x,y), and for any $S \neq \emptyset, E$, $d_S^{B_j}(x,y) = 0$ if $S \ni j$, $d_S^{B_j}(x,y) = 1$ otherwise.

Corollary 1 *For all x,y,z, if there do not exist $\mathbf{p_1},\mathbf{p_2},\mathbf{p_3}$ in Ω with*

$$x\mathbf{p_1}z\mathbf{p_1}y,\ x\mathbf{p_2}y\mathbf{p_2}z,\ z\mathbf{p_3}x\mathbf{p_3}y,$$

then B_j is a non-dictatorial n-person ASWF for every j and $n > 2$.

Proof. It is clear that by definition, d^{B_j} satisfies (1, 2). To see that it satisfies (3), observe that in every partition of the agents, one of the sets obtained must contain j. Say $j \in A\cup U\cup V$. If $d_{A\cup U\cup V}^{B_j}(x,y) = 0$, then (3) is clearly valid. So we may assume that $d_{A\cup U\cup V}^{B_j}(x,y) = 1$. This happens only when $A \cup U \cup V = E$ (or if (x,y) is trivial, which in turns imply that all the other sets are empty). We may assume $U, V \neq \emptyset$ and $j \in A$, otherwise (3) is clearly valid. But according to condition (*), this implies existence of $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω with

$$x\mathbf{p_1}z\mathbf{p_1}y,\ x\mathbf{p_2}y\mathbf{p_2}z,\ z\mathbf{p_3}x\mathbf{p_3}y,$$

which is a contradiction.

So, d^{B_j} satisfies (1-3) and hence corresponds to an ASWF. When $n > 2$, B_j is clearly non-dictatorial. ∎

2.2 Single-peaked preferences for the n-person case

The class of single-peaked preferences has received a great deal of attention. The domain Ω is single-peaked with respect to a linear ordering $\mathbf{q}$ over A if $\Omega \subseteq \{\mathbf{p} \in \Sigma :$ for every triple (x, y, z) if $x\mathbf{q}y\mathbf{q}z$ then it is **not** the case that $x\mathbf{p}y$ and $z\mathbf{p}y\}$.

The 0-1 solutions to the IP (1–3) correspond to Arrovian Social Welfare functions. We prove that for single-peaked domains, the constraints (1–3) are *sufficient* to characterize the convex hull of the 0-1 solutions.

Theorem 2 *When Ω is single-peaked the set of non-negative solutions satisfying (1-3) is an integral polytope. All ASWF's are extreme point solutions of this polytope.*

Proof. To see this, we prove that every fractional solution satisfying (1-3) can be written as a convex combination of 0-1 solutions satisfying the same set of constraints. Let $\mathbf{q}$ be the linear ordering with respect to which Ω is single-peaked.

Rounding Scheme:
Let $d_S(\cdot)$ be a (possibly) fractional solution to the LP. We round the solution d to the 0-1 solution d' in the following way:

- Generate a random number Z uniformly between 0 and 1.
- For $a, b \in A$ with $a\mathbf{q}b$, and $S \subset E$, then
  - $d'_S(a, b) = 1$, if $d_S(a, b) > Z$, 0 otherwise;
  - $d'_S(b, a) = 1$, if $d_S(b, a) \geq 1 - Z$, 0 otherwise.

The integral solution obtained is feasible:
The 0-1 solution d'_S generated in the above manner clearly satisfies constraints (1). To see that it also satisfies constraint (2), for any a, b, we need to show that $d'_T(a, b) + d'_{T^c}(b, a) = 1$, for any $T \subseteq E$. Suppose not. Then $d'_T(a, b) = d'_{T^c}(b, a) = 1$ or $d'_T(a, b) = d'_{T^c}(b, a) = 0$. Without loss of generality, let $a\mathbf{q}b$. If $d'_T(a, b) = d'_{T^c}(b, a) = 1$, we have $d_T(a, b) > Z$ and $d_{T^c}(b, a) \geq 1 - Z$, which contradicts the constraint $d_T(a, b) + d_{T^c}(b, a) = 1$. If, on the other

hand, we have $d'_T(a,b) = d'_{T^c}(b,a) = 0$, $d_T(a,b) \leq Z$ and $d_{T^c}(b,a) < 1 - Z$, which also contradicts the constraint $d_T(a,b) + d_{T^c}(b,a) = 1$.

We show next that all the constraints in (3) are satisfied by the solution $d'_S(\cdot)$. Consider three alternatives a, b, c, and constraint (3) (with a, b, c replacing the role of x, y, z) can be re-written as:

$$d_{A \cup U \cup V}(a,b) + d_{B \cup U \cup W}(b,c) + d_{C \cup V \cup W}(c,a) \leq 2.$$

Suppose $a\mathbf{q}b\mathbf{q}c$. Then in constraints (3), by the single-peakedness property, we must have $A = V = \emptyset$. In this case, the constraint reduces to

$$d_U(a,b) + d_{B \cup U \cup W}(b,c) + d_{C \cup W}(c,a) \leq 2.$$

We need to show that

$$d'_U(a,b) + d'_{B \cup U \cup W}(b,c) + d'_{C \cup W}(c,a) \leq 2.$$

By choosing the sets in constraints (3) in a different way, with

$$U' \leftarrow U, B' \leftarrow B, W' \leftarrow W \cup C, C' \leftarrow \emptyset,$$

we have a new inequality

$$d_{U'}(a,b) + d_{B' \cup U' \cup W'}(b,c) + d_{C' \cup W'}(c,a) \leq 2,$$

which is equivalent to

$$d_U(a,b) + 1 + d_{C \cup W}(c,a) \leq 2.$$

Hence we must have $d_U(a,b) + d_{C \cup W}(c,a) \leq 1$. Note that since $a\mathbf{q}b$ and $b\mathbf{q}c$, our rounding scheme ensures that $d'_U(a,b) + d'_{C \cup W}(c,a) \leq 1$. Hence

$$d'_U(a,b) + d'_{B \cup U \cup W}(b,c) + d'_{C \cup W}(c,a) \leq 2.$$

To finish the proof, we need to show that constraint (3) holds for different orderings of a, b and c under $\mathbf{q}$; the above argument can be easily extended to handle all these cases to show that constraint (3) is valid. We omit the details here.

<u>All extreme point solutions are integral:</u>
Suppose now that the fractional solution d_S is a fractional extreme point in the polytope defined by constraints (1-3). By standard polyhedra theory, there exists a cost function $(c_S(a,b))$ such that d_S is the unique minimum solution to the problem:

$$\begin{array}{rl} (P) \quad \min & \sum_{S,a,b} c_S(a,b) x_S(a,b) \\ \text{subject to :} & x_S(a,b) \text{ satisfies constraints (1-3);} \\ & x_S(a,b) \in [0,1]. \end{array}$$

The rounding scheme we have just described converts a fractional solution to a 0-1 solution satisfying

$$E(d'_S(a,b)) = P(Z < d_S(a,b)) = d_S(a,b), \text{ if } a\mathbf{q}b,$$

and

$$E(d'_S(a,b)) = P(Z \geq 1 - d_S(a,b)) = d_S(a,b), \text{ if } b\mathbf{q}a.$$

Hence $E(d'_S(a,b)) = d_S(a,b)$ for all S, a and b. Thus

$$E(\sum_{S,a,b} c_S(a,b)d'_S(a,b)) = \sum_{S,a,b} c_S(a,b)d_S(a,b),$$

and hence all the 0-1 solutions obtained by the rounding scheme must also be a minimum solution to problem (P). This contradicts the fact that d_S is the unique minimum solution.

■

The argument above shows the set of ASWF's on single-peaked domains (wrt $\mathbf{q}$) has a property similar to the generalized median property of the stable marriage problem (see Teo and Sethuraman [18]):

Theorem 3 *Let $f_1, f_2, \ldots, f_N$ be distinct ASWF's for the single-peaked domain Ω (with respect to $\mathbf{q}$). Define a function $F_k : \Omega^n \to \Sigma$ with the property:*

The set S under F_k is decisive for x over y if

$x\mathbf{q}y$, and S is decisive for x over y for at least $k+1$ *of the ASWF f_i's; or*

$y\mathbf{q}x$, and S is decisive for x over y for at least $N-k$ *of the ASWF f_i's.*

Then F_k is also an ASWF.

One consequence of Theorem 3 is that when Ω is single-peaked, it is Arrovian, since the dictatorial ASWF can be used to construct non-dictatorial ASWF in the above manner.

3 2-person ASWF's

From Kalai and Muller [7] we know that **every** 2-person ASWF corresponds to a feasible 0-1 solution to the system (1, 2, 4–7) and vice-versa.

Two obviously feasible solutions are $d_1(x,y) = 1, d_2(x,y) = 0, \forall (x,y) \in A^2$ and $d_1(x,y) = 0, d_2(x,y) = 1 \forall (x,y) \in A^2$. The first corresponds to an ASWF in which agent 1 is the dictator and the second, by default, to one in which agent 2 is the dictator. We refer to these solutions as the all 1's solution and all the 0's solution respectively.

The main result of [7] can be phrased as follows: a non-dictatorial solution to (1, 2, 4 - 7) exists for the case $n = 2$ agents if and only if a non-dictatorial solution to (1-3) exists for any n. We show how this result can be proved using the IP formulation.

It is easy to show that a non-dictatorial solution to (1, 2, 4–7) (for $n = 2$) automatically implies a non-dictatorial solution to (1–3). For a proper subset, S, of agents, let

$$d_S(x,y) = \begin{cases} d_1(x,y) & \text{if } 1 \in S, \\ d_2(x,y) & \text{otherwise.} \end{cases}$$

It is easy to see that the function d defined above corresponds to an n-person ASWF, and hence constraints (1)–(3) are satisfied. Next, we give a proof of the converse.

Let d^* be a non-dictatorial solution to (1-3). If there is a set of agents S such that $d^*_S(x,y)$ is non-zero for some but not all $(x,y) \in A^2$, we are done because $d_1 = d^*_S, d_2 = d^*_{S^c}$ would be a non-dictatorial solution to (1, 2, 4–7). Thus we assume that for each $S \subseteq E$, $d^*_S(x,y)$ is either all 0 or all 1, for all $(x,y) \in A^2$.

Case 1: Suppose the born loser rule B_j, $j \neq 1,2$ is a non-dictatorial ASWF for the domain Ω. We can construct a non-dictatorial 2-person ASWF in the following manner:

- Assign a fixed preference $\mathbf{p}$ to j.
- For $i = 1,2$, $d_i(x,y) = 1$ if and only if $y\mathbf{p}x$.
- $d_{1,2}(x,y) = 1$ for all $x,y \in A^2$.

The born loser rule thus induces a non-dictatorial ASWF for $n = 2$.

Case 2: We may assume that there exists $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω with

$$x\mathbf{p_1}z\mathbf{p_1}y, \; x\mathbf{p_2}y\mathbf{p_2}z, \; z\mathbf{p_3}x\mathbf{p_3}y.$$

This rules out the existence of sets S,T in E such that $d_S = d_T = 1$, $S \cap T = \emptyset$, otherwise constraint (3) will be violated for the triple x,y,z.

Observe that for each agent j, $d_j(x,y)$ is either all 0 or all 1, for all $(x,y) \in A^2$. One of two things must be true:

1. $\exists\, j$ such that $d^*_j(x,y) = 1$ for all $(x,y) \in A^2$, or
2. $d^*_j(x,y) = 0$ for all j and $(x,y) \in A^2$.

Suppose the first holds. Since d^* is non-dictatorial, there is a subset $S \ni j$ such that $d^*_S(x,y) = 0$ for some $(x,y) \in A^2$, otherwise j would be a dictator. Now j, S^c are disjoint sets with $d_j(x,y) = d_{S^c}(x,y) = 1 \; \forall (x,y) \in A^2$, which cannot be possible by our assumption.

Suppose now that $d^*_j(x,y) = 0$ for all j and $(x,y) \in A^2$. By equation (1) we can choose a set T such that $d^*_T(x,y) = 1 \; \forall (x,y) \in A^2$. Suppose also that T is minimal with respect to this condition. Choose any $j \in T$. By the minimality condition on T it follows that $d_{T \setminus j}(x,y) = 0$ and $d_{T^c \cup j}(x,y) = 1$ for all $(x,y) \in A^2$. By equation (1) we have that $d_{T^c}(x,y) = 0 \; \forall (x,y) \in A^2$. Since $d_j(x,y) = 0$ for all $(x,y) \in A^2$ it follows that $d_{T \setminus j \cup T^c}(x,y) = 1$ for all $(x,y) \in A^2$. We now define a 3 agent ASWF. Label the 3 agents $\{1,2,3\}$.

1. $d'_i(x,y) = 0$ for all $i = 1,2,3$ and $(x,y) \in A^2$,

2. $d'_{1,2}(x,y) = d_{T\setminus j \cup j}(x,y) = 1$ for all $(x,y) \in A^2$,

3. $d'_{1,3}(x,y) = d_{T\setminus j \cup T^c}(x,y) = 1$ for all $(x,y) \in A^2$,

4. $d'_{2,3}(x,y) = d_{j \cup T^c}(x,y) = 1$ for all $(x,y) \in A^2$.

The function we have just defined, d', is 3-person majority rule. Since inequality (3) is satisfied by the d, it follows that d' does not 'cycle' on Ω. Thus d' is an ASWF on Ω. We now construct a 2-person ASWF using d'. Fix any $\mathbf{p} \in \Omega$. Now apply 3-person majority rule to the 2 agent society with a dummy third agent whose preference ordering is always $\mathbf{p}$.

We have thus shown that given an n-person non-dictatorial ASWF, we can always construct a 2-person non-dictatorial ASWF. ■

4 Sufficient Conditions

We now use the system (1, 2, 4-7) to derive the main impossibility results about Arrovian domains. In view of the Kalai-Muller result we can restrict our attention to 2-person ASWF's.

It is instructive to ignore, for the moment, inequalities of types (6) and (7). The constraint matrix associated with the inequalities of types (1, 2, 4, 5) and $0 \leq d(x,y) \leq 1$ $\forall (x,y) \in A^2$ is totally unimodular. This is because each inequality can be reduced to one that contains at most two coefficients of opposite sign and absolute value of 1.[2] Hence the extreme points are all integral, in fact 0-1. If one or more of these extreme points was different from the all 0's solution and all 1's solution we would know that Ω is Arrovian. If the only extreme points were the all 0's solution and all 1's solution that would imply that Ω is not Arrovian.

Thus difficulties with determining the existence of a feasible 0-1 solution different from the all 0's and all 1's solution have to do with the inequalities of the form (6) and (7). Notice that any admissible ordering (by Ω) of three alternatives gives rise to an inequality of types (6) and (7). However some of them will be redundant. Constraints (6, 7) are not redundant only when they are obtained from a triplet (x,y,z) with the property:

There exists $\mathbf{p}$ such that $x\mathbf{p}y\mathbf{p}z$ but no $\mathbf{q} \in \Omega$ such that $y\mathbf{q}z\mathbf{q}x$ or $z\mathbf{q}x\mathbf{q}y$.

Such a triplet is called an *isolated triplet.*

Call the inequality representation of Ω, by inequalities of types (1, 2, 4, 5), the **unimodular representation** of Ω. Note that all inequalities in the unimodular representation are of the type $d(x,u) \leq d(x,v)$ or $d(u,x) \leq d(v,x)$. *Furthermore,* $d(x,u) \leq d(x,v)$ *and* $d(u,y) \leq d(v,y)$ *appear in the representation only if there exist* $\mathbf{p},\mathbf{q}$ *with* $u\mathbf{p}x$ *and* $v\mathbf{p}x$ and $x\mathbf{q}u$ *and* $x\mathbf{q}v$.

This connection allows us to provide a graph-theoretic representation of the unimodular representation of Ω as well as a graph-theoretic interpretation of when Ω is not Arrovian.

[2] It is well known that such matrices are totally unimodular. See for example, Theorem 11.12 in [1].

With each *non-trivial* element of A^2 we associate a vertex. If in the unimodular representation of Ω there is an inequality of the form $d_1(a,b) \leq d_1(x,y)$ where (a,b) and $(x,y) \in A^2$ then insert a *directed* edge from (a,b) to (x,y). Call the resulting directed graph D^{Ω}.

If (x,y) is a trivial pair (and hence $(x,y) \notin D^{\Omega}$), then $d_1(x,y)$ is automatically fixed at 1, and $d_1(y,x)$ fixed at 0. An inequality of the form $d_1(x,y) \leq d_1(x,z)$ (or $d_1(z,y)$) *cannot* appear in the unimodular representation, for any alternative z in A. Otherwise there must be some $\mathbf{p} \in \Omega$ with $y\mathbf{p}x$. Similarly, if (x,y) is trivial, $d_1(y,x) \geq d_1(z,x)$ (or $d_1(y,z)$) *cannot* appear in the unimodular representation, for any alternative z in A. Thus fixing the values of $d_1(x,y)$ and $d_1(y,x)$ arising from a trivial pair (x,y) does not affect the value of $d_1(a,b)$ for $(a,b) \in D^{\Omega}$.

A subset S of vertices in D^{Ω} is **closed** if there is no edge directed out of S. That is, there is no directed edge with its tail incident to a vertex in S and its head incident to a vertex outside S. Notice that $d_1(x,y) = 1\ \forall (x,y) \in S$ and 0 otherwise (and together with those arising from the trivial pairs) is a feasible 0-1 solution to the unimodular representation of Ω if S is closed. Hence every closed set in D^{Ω} corresponds to a feasible 0-1 solution to the unimodular representation. The converse is also true.

Theorem 4 *If D^{Ω} is strongly connected then Ω is non-Arrovian.*

Proof. The set of all vertices of D^{Ω} is clearly a closed set. The solution corresponding to this closed set is the ASWF where agent 1 is the dictator. The empty set of vertices is closed and this corresponds to agent 2 being the dictator. Hence, Ω is Arrovian if and only D^{Ω} contains a closed, non-empty proper subset of vertices. However D^{Ω} has a closed, non-empty proper subset of vertices if and only if D^{Ω} is not strongly connected.[3] ■

We note that verifying that a directed graph is strongly connected can be done efficiently. See [1] for details.

4.1 Application 1: Arrow's Theorem

An obvious application is to Arrow's impossibility theorem. In this case $\Omega = \Sigma$ and we need to verify that between any ordered pair of vertices of D^{Ω} there is a directed path from one to the other. Each vertex corresponds to a non-trivial pair. Let the pairs corresponding to these two vertices be (x,y) and (u,v). Since $\Omega = \Sigma$, all possible orderings of the following triples are possible: $\{x,y,u\}, \{x,y,v\}, \{x,u,v\}, \{y,u,v\}$. In particular from inequalities (4) and (5) we get $d(x,y) \leq d(x,u)$ and $d(x,u) \leq d(v,u)$, and so there is a path $(x,y) \to (x,u) \to (u,v)$. Since the choice of (x,y) and (u,v) was arbitrary, it follows that D^{Ω} is strongly connected.

4.2 Application 2: Saturating Preference Domains

To describe the next application, we need a few additional definitions. A triple $x,y,z \in A$ is called **free** if each possible ordering of the triple is attained by some permutation in Ω. In

[3]A directed graph is strongly connected if there is a directed cycle through every pair of vertices.

other words, a triple $x, y, z \in A$ is free, if the preferences of any individual are unrestricted on $\{x, y, z\}$. Two pairs α and β contained in A are called **strongly connected** if $\alpha \cup \beta$ is a free triple.[4] They are called connected if there exists a finite sequence of pairs $\lambda(1), \lambda(2), ..., \lambda(k)$ contained in A with $\lambda(1) = \alpha, \lambda(k) = \beta$, and $\lambda(j) \cup \lambda(j+1)$ is strongly connected for all $j = 1, 2, ..., k-1$. Ω is called **saturating** if

1. there exists at least 2 non-trivial pairs, and
2. any two non-trivial pairs are connected.

Kalai, Muller and Satterthwaite [8] show that if Ω is saturating, then Ω is non-arrovian. This important sufficient condition is the basis of many impossibility results in the social choice literature. We refer the reader to Le Breton and Weymark [14] for a comprehensive survey. The saturating criterion is not a necessary condition for dictatorship. It is immediate from the definition of saturating preference domains that D^{Ω} is strongly connected and so non-arrovian.

4.3 Application 3: Linked Domains

As another application we consider a restriction on Ω introduced in [6]: Ω is **minimally rich** if for every $x \in A$ there is a $\mathbf{p} \in \Omega$ which ranks x first. The main result in [6] is that a subclass of minimally rich domains which they call **linked** does not admit the existence of an onto strategy proof social choice function.

Given a $\mathbf{p} \in \Omega$, denote the k^{th} ranked alternative in $\mathbf{p}$ by $r^k(\mathbf{p})$. A set $K \subset A$ of at least three alternatives is called **internally complete** if for all ordered pairs (x, y) in K there exists $\mathbf{p} \in \Omega$ such that $r^1(\mathbf{p}) = x$ and $r^2(\mathbf{p}) = y$. In particular, this implies that all possible orderings of the alternatives in K are found in Ω.

Two sets of alternatives K and M, each with at least three alternatives, are *adjacent* if there is an $x \in K$ which is ranked second in two orderings headed by distinct elements of M and a y in M which is ranked second in two orderings headed by distinct elements of K. Consider the graph, L^{Ω} with a vertex corresponding to each internally complete set, and edges between pairs of vertices corresponding to adjacent internally complete sets. The domain Ω is linked if the following two statements hold.

- The elements of A can be partitioned into a collection $K_1, \ldots, K_s$ of internally complete sets; and
- The associated graph, L^{Ω}, is connected, i.e. there is a path between every pair of vertices.

Notice that a linked domain is minimally rich. It is also easy to see that if Ω is linked then D^{Ω} is strongly connected.

[4]The term strongly connected here has a different meaning to the term strongly connected in a directed graph.

4.4 Application 4: Cyclic Domains

Kim and Roush (1980) consider the domain Ω consisting of all cyclic permutations of a fixed ordering of the alternatives. In this domain it is easy to see that in the system (1, 2, 4-7), the inequalities (6) and (7) are redundant. So, in this case the domain is dictatorial if and only if D^{Ω} is strongly connected. It is easy to verify that in this instance D^{Ω} is not strongly connected and so Ω admits a non-dictatorial ASWF.

5 Complexity and Polyhedral Structure

To study the complexity of deciding whether a domain is Arrovian, we must first specify how Ω is encoded. If Ω has exponentially many permutations (say $O(2^n)$) elements, where $n = |A|$), then the straight forward input model needs at least $O(2^n)$ bits. Note that the number of the decision variables for the Integer Program for 2-person ASWF's is polynomial in n. Furthermore, the time complexity of verifying the existence of triplets in Ω can be trivially performed in time $O(n^3 2^n)$. Hence the decision version of the existence of ASWF can be solved in time polynomial in the size of the input.

Suppose, however, instead of listing the elements of Ω, we prescribe a polynomial time oracle to check membership in Ω. The complexity issue of deciding whether the domain is Arrovian now depends on how we encode the membership oracle, and not on the number of elements in Ω. In this model, we exhibit an example to show that checking whether a triplet exists in Ω is already NP-hard.

Consider a domain Ω where membership is specified by a set of forbidden triplets (denoted by Δ). To check whether p is in Ω, we only need to verify that p does not contain any triplet in the set Δ. Thus membership can be checked in polynomial time.

Claim 3 *For any given triplet (x, y, z), it is NP-complete to determine whether there exists $\mathbf{p} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$.*

Proof: This follows from the NP-completeness of the *Betweenness Problem* (See Opatrny [13] or Chor and Sudan [4]), which is as follows:

> Given a set of betweenness constraints on triplets (in the form y_k is between x_k and z_k, $k = 1, \dots, m$), is there a permutation that satisfies all the specified betweenness constraints?

To see the connection between the two problems, note that the betweenness constraints can be viewed as conditions on triplets. The constraint y_k is between x_k and z_k can be viewed as constraints of the form: $\mathbf{p}$ satisfies the betweenness condition if it is not the case that $x_k\mathbf{p}y_k$ and $z_k\mathbf{p}y_k$ or $y_k\mathbf{p}x_k$ and $y_k\mathbf{p}z_k$. If we can answer the first problem in polynomial time, then given any domain Ω specified by forbidden triplets, and any triplet (x, y, z), we can check in polynomial time whether there exists $p \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$. We can use this routine recursively to determine whether there exists a permutation that satisfies all the betweenness constraints in the following manner:

1. $N \leftarrow 1$.

2. Let Ω_N be specified by forbidden triplets of the type

$$\{x_k \mathbf{p} z_k \mathbf{p} y_k, z_k \mathbf{p} x_k \mathbf{p} y_k, y_k \mathbf{p} z_k \mathbf{p} x_k, y_k \mathbf{p} x_k \mathbf{p} z_k : k = 1, \ldots, N\}.$$

3. Check whether there exists $p \in \Omega_N$ such that $x_{N+1} \mathbf{p} y_{N+1} \mathbf{p} z_{N+1}$ or $z_{N+1} \mathbf{p} y_{N+1} \mathbf{p} x_{N+1}$. If $N \leq m$ and no such $\mathbf{p}$ exists in Ω_N, then the answer to the betweenness problem is "No". Otherwise $N \leftarrow N + 1$.

4. If $N \leq m$, go to Step 2. Otherwise, we conclude that the answer to the betweenness problem is "YES".

Note that Step 3 is executed at most $O(n^3)$ times and hence if Step 3 can be executed in polynomial time, this will give rise to a polynomial routine for the betweenness problem which is known to be NP-complete. ■

Hence given an Ω specified by forbidden triplets, it is NP-hard just to write down the set of inequalities required by our integer program!

The above result yields evidence that given a domain Ω, it is in general very difficult to check whether Ω is an Arrovian domain. In the rest of this section, we investigate the polyhedral structure of the polyhedron defined by (1–3). We restrict our attention to problems involving just two agents, conveniently labelled 1 and 2; also we can consider the equivalent class of inequalities defined by (1, 2, 4–7). For any pair of alternatives a and b, we know, by (2), that

$$d_2(a, b) \;=\; 1 - d_1(a, b),$$

using which we can eliminate all of the variables corresponding to agent 2. Since all of the variables correspond to agent 1, we suppress the subscript hereafter, and simply use $d(a, b)$ instead of $d_1(a, b)$.

We describe a sequential lifting method to derive valid inequalities for the problem, using the directed graph D^Ω. We say that the node u *dominates* the node v if there is a directed path in D^Ω from v to u (i.e. $d(u) \geq d(v)$).

Sequential Lifting Method:

- For each isolated triplet (x, y, z), we have the inequality

$$1 + d(x, z) \geq d(x, y) + d(y, z). \tag{8}$$

- Let $D(x, y)$ (and resp. $D(y, z)$) denote the set of nodes in D^Ω that is dominated by the node (x, y) (resp. (y, z)) in D^Ω.

- For each node (a,b) in D^{Ω}, if

$$u \in D(a,b) \cap D(x,y) \neq \emptyset,\ v \in D(a,b) \cap D(y,z) \neq \emptyset,$$

then the constraint arising from the isolated triplet can be augmented by the following valid inequalities:

$$d(a,b) + d(x,z) \geq d(u) + d(v). \tag{9}$$

To see the validity of the above constraint, note that by the definition of domination, we have

$$d(x,y) \geq d(u), d(y,z) \geq d(v), d(a,b) \geq d(u), d(a,b) \geq d(v).$$

If $d(a,b) = 0$, then $d(u) = d(v) = 0$ and hence (9) is trivially true. If $d(a,b) = 1$, then (9) follows from (8).

5.1 Example with Three Alternatives

We first show that the polyhedron defined by (1, 2, 4-7) need not be integral using a simple example. Let $A = \{x,y,z\}$, and let

$$\Omega = \{xyz, yzx, zxy, xzy\}.$$

From (4–7), we get the following system of inequalities:

$$\begin{gathered}
d(x,y) \leq d(x,z), \\
d(z,x) \leq d(y,x), \\
d(y,z) \leq d(y,x), \\
d(x,y) \leq d(z,y), \\
d(z,x) \leq d(z,y), \\
d(y,z) \leq d(x,z), \\
d(x,z) + d(z,y) \leq 1 + d(x,y), \\
d(y,z) + d(z,x) \geq d(y,x).
\end{gathered}$$

A fractional extreme point of this system is

$$d(x,z) = d(y,x) = d(y,z) = d(z,x) = d(z,y) = 0.5;\ \ d(x,y) = 0.$$

The only other fractional extreme point is:

$$d(x,y) = d(x,z) = d(y,z) = d(z,x) = d(z,y) = 0.5;\ \ d(y,x) = 1.$$

We next use the sequential lifting method to identify new valid inequalities from the isolated triplet (x, z, y).

Consider the following set of inequalities:

$$\begin{aligned} 1 + d(x, y) &\geq d(x, z) + d(z, y) \\ &\geq d(y, z) + d(z, x). \end{aligned} \tag{10}$$

Note that (x, z) dominates (y, z), and (z, y) dominates (z, x). We also have $d(y, x) \geq d(y, z)$, and $d(y, x) \geq d(z, x)$, and hence (y, x) dominates both (y, z) and (z, x). The sequential lifting method gives rise to

$$d(x, y) + d(y, x) \geq d(y, z) + d(z, x). \tag{11}$$

Also, for a pair of alternatives a and b, replacing $d(a, b)$ with $1 - d(b, a)$, results in another valid inequality, which we record as

$$d(x, y) + d(y, x) \leq d(z, y) + d(x, z). \tag{12}$$

More importantly, Eqs. (11) and (12) are *facets*. To see this, we first observe that the underlying polyhedron is full-dimensional (dimension 6); and its extreme points are

$$\Big\{ e_1 = (0,0,0,0,0,0),\ e_2 = (0,0,0,0,0,1),\ e_3 = (0,0,1,0,1,1),$$
$$e_4 = (0,1,0,0,0,0),\ e_5 = (0,1,1,1,0,0),\ e_6 = (1,1,0,0,0,1),$$
$$e_7 = (1,1,1,0,1,1),\ e_8 = (1,1,1,1,0,1),\ e_9 = (1,1,1,1,1,1) \Big\},$$

where the components of each entry represent $d(x, y)$, $d(x, z)$, $d(y, x)$, $d(y, z)$, $d(z, x)$, and $d(z, y)$ (in that order). The elements e_1, e_2, e_3, e_4, e_5 and e_9 are affinely independent, and satisfy (11) as an equality; the elements e_1, e_3, e_5, e_7, e_8, and e_9 are affinely independent, and satisfy (12) as an equality. These two observations show, respectively, that Eqs. (11) and (12) are facets.

For $|A| = 3$, we enumerate all possible domains, and observe that the strengthened formulation using the sequential lifting method defines the convex hull of all ASWF's in each case.

5.2 Examples with Four Alternatives

Preliminary observations. For $|A| = 4$, the number of possible domains, Ω, is 2^{24}; of these, we ignore domains that contain at most 1 alternative, which leaves us $2^{24} - 25$ possibilities to consider. From the IP formulation, however, we know that two domains that generate the same set of "triplets," are either both dictatorial or both non-dictatorial; so the number of possibilities to be examined depends only on the number of different sets of triplets generated by the domains. This observation reduces the number of possibilities to

77850, which is substantially smaller than all potential subsets of triples. In addition, we can further restrict the possibilities to be explored using "symmetries:" Two distinct collections of triplets are isomorphic if one collection can be obtained from the other by simply renaming the alternatives. Clearly, if a collection of triplets (generated by some domain) is dictatorial, so are all of its isomorphic equivalents. The number of distinct collections of triplets that are not isomorphic to one another is 3315.

LP/IP relationship. For $|A| = 3$, we observed that whenever the LP relaxation of the original formulation (1, 2, 4–7) has a non-trivial (possibly fractional) solution, so does the corresponding IP. This raises the possibility that although finding a compact description of the set of all SWFs may be difficult, or even impossible, we might still be able to resolve the existence/non-existence of an SWF by solving the associated LP relaxation; such a result is true for the stable roommates problem; see [18]. The following example, however, rules out such a possibility.

Example 1: Let

$$\Omega = \{wyzx, wzxy, xywz, xzyw, ywzx, yzxw, zwxy\}.$$

The associated set of triplets is

$$\begin{aligned} T = \{&wxy, wyx, xyw, ywx, yxw, wzx, xwz, xzw, zwx, zxw \\ &wyz, wzy, ywz, yzw, zwy, zyw, xyz, xzy, yzx, zxy\} \end{aligned}$$

It is easily verified that all the decision variables except $d(x, y)$ and $d(y, x)$ are equal to one another in every feasible LP solution; this is a consequence of the basic formulation (1, 2, 4–7). The fractional LP solution

$$d(x, y) = 0,\ d(y, x) = 1; \quad \text{all other variables} = 0.5,$$

is feasible. If the variables are restricted to be 0-1, it is easy to verify that Ω is dictatorial.

Additional valid inequalities. Consider the domain

$$\Omega = \{wxyz, wxzy, wzyx, zwyx, zxyw, zywx\},$$

with the associated set of triplets being

$$\begin{aligned} T = \{&wxy, xyw, ywx, wyx, zwx, wxz, wzx, zxw, \\ &wzy, zyw, zwy, wyz, zxy, xyz, xzy, zyx\} \end{aligned}$$

In T, the triplet wyx is the only isolated triplet; if it were absent, the LP relaxation of the IP associated with T would be exact.

As before, we try to strengthen the formulation by finding additional valid inequalities using the sequential lifting method. To that end, consider the following set of inequalities, each of whose justification is included alongside, in parenthesis:

$$\begin{aligned}
d(w,x)+1 &\geq d(y,x)+d(w,y), \quad \text{(isolated triplet } wyx\text{)} && (13)\\
d(w,y) &\geq d(x,y), \quad \text{(by } \{ywx, wxy\}\text{)} && (14)\\
d(z,x) &\geq d(z,w), \quad \text{(by } \{zwx, wxz\}\text{)} && (15)\\
d(z,w) &\geq d(x,w), \quad \text{(by } \{wzx, zxw\}\text{)} && (16)\\
d(x,w) &\geq d(x,y), \quad \text{(by } \{xyw, ywx\}\text{)} && (17)\\
d(z,x) &\geq d(y,x). \quad \text{(by } \{xzy, zyx\}\text{)} && (18)
\end{aligned}$$

From Eqs. (15)-(17), we have

$$d(z,x) \geq d(x,y). \tag{19}$$

From Eqs. (13) and (14), we have

$$d(w,x)+1 \geq d(y,x)+d(x,y), \tag{20}$$

which together with Eqs. (18) and (19) imply

$$d(w,x)+d(z,x) \geq d(y,x)+d(x,y). \tag{21}$$

As before, replacing $d(a,b)$ with $1-d(b,a)$, results in another valid inequality, which we record as

$$d(x,w)+d(x,z) \leq d(x,y)+d(y,x). \tag{22}$$

It is easy, but tedious, to verify that inequalities (21) and (22) are facets of the underlying polyhedron. It is also interesting to note that these are the first inequalities involving *four* alternatives.

When $|A|=4$, we observe that, for each domain, the LP formulation, augmented with inequalities constructed using the sequential lifting method, whenever applicable, defines the convex hull of all ASWF's. A natural question is if whether the sequential lifting method will gives rise to all facets even for the case $|A| \geq 5$; we do not yet know, although we suspect the answer to be negative.

6 Majority Rule

In the universe of preference aggregation procedures, (simple) majority rule occupies a special place. While majority rule satisfies unanimity and the independence of irrelevant alternatives condition it is not guaranteed to produce a transitive ordering. In other words majority rule is not an ASWF for all Ω.

Here we show how the integer program can be used to derive, in a simple way, two of the well known results about majority rule. The first is a characterization of those Ω on which majority rule is an ASWF. This result is due essentially to Sen (1966). We assume, so that majority rule is well defined, that the number of agents is odd.

Recall that Ω admits a *Condorcet* triple if there are x, y and $z \in A$ and $\mathbf{p_1}$, $\mathbf{p_2}$ and $\mathbf{p_3} \in \Omega$ such that $x\mathbf{p_1}y\mathbf{p_1}z$, $y\mathbf{p_2}z\mathbf{p_2}x$, and $z\mathbf{p_3}x\mathbf{p_3}y$.

Theorem 5 *Majority rule is an ASWF on Ω if and only if Ω does not contain a Condorcet triple.*

Proof

Suppose first that majority rule is an ASWF on Ω. To get a contradiction assume that $x, y, z \in A$ form a Condorcet triple. Let n, the number of agents, be $3r + k$ where $r \geq 1$ is integral and $0 \leq k \leq 2$ is also integral.

If $k = 0$, partition the agents into three sets of size r called U, V and W. Every agent in U ranks x above y above z. Every agent in V ranks z above x above y. Every agent in W ranks y above z above x. Since n is odd and $2r > n/2$ it follows that on this profile that majority rule produces a cycle.

If $k = 1$, choose U, V and W as above but $|U| = |V| = r$ and $|W| = r + 1$. Once again $2r + 1 > 2r > n/2$, so majority rules cycles again. If $k = 2$ repeat the argument with $|U| = r$ and $|V| = |W| = r + 1$.

Now suppose that Ω has no Condorcet triple. To show that majority rule is an ASWF we must show that inequality (3) is satisfied. To obtain a contradiction suppose not and fix a triple $x, y, z \in A$ for which (3) is violated. Since Ω has no Condorcet triple, at least one of A, B or C is empty and at least one of U, V and W is empty. Without loss of generality suppose that $A, W = \emptyset$. Since (3) is violated we have $d_{U\cup V}(x, y) = d_{B\cup U}(y, z) = d_{C\cup V}(z, x) = 1$ (a similar argument applies in the case when they are all zero). Majority rule implies that $|B| + |U| > n/2$ and $|C| + |V| > n/2$. Adding these two inequalities produces:

$$n = |B| + |U| + |C| + |V| > n,$$

a contradiction. ■

The next theorem we take up is due to Maskin (1995). An ASWF is called **anonymous** if its ranking over pairs of alternatives remains unchanged when the labels of the agents are permuted. Hence $d_S(x, y)$ depends only on $|S|$ for all $(x, y) \in A^2$. In particular a dictatorial rule is not anonymous. An ASWF is called **neutral** if its ranking over any pair of alternatives depends only on the pattern of voters' preferences over that pair, not on the alternatives' labels. It is stronger than independence of irrelevant alternatives. Neutrality implies that $d_S(x, y) = d_S(a, b)$ for any $(x, y), (a, b) \in A^2$. Thus the value of $d_S(\cdot, \cdot)$ is determined by $|S|$ alone. In the proof below we use this observation to write d_S as d_r where $r = |S|$.

It is easy to see that majority rule is both anonymous and neutral but is not the only such rule. For three agents, three person minority rule is anonymous and neutral. In three person minority rule only singleton sets and the entire set of agents is decisive.

Theorem 6 *Let Ω admit an anonymous, neutral ASWF. Then majority rule is an ASWF on Ω.*

Proof

From the previous result it suffices to show that Ω has no Condorcet triples. To obtain a contradiction suppose not. Let $x, y, z \in A$ be a collection that forms a Condorcet triple. Let d be a non-dictatorial, anonymous, neutral ASWF. Thus d satisfies (1,2,3). Let n denote the number of voters.

Inequality (3) implies:

$$1 \le d_a(x,y) + d_b(y,z) + d_c(z,x) \le 2,$$

whenever

$$a, b, c > 0, a + b + c = n.$$

Note that by neutrality, $d_S(x,y) = d_S(a,b)$ for all $(x,y), (a,b)$. So we omit the alternatives and represent the variables as d_S.

Note that by (2), $d_1 \neq d_{n-1}$. Furthermore, since

$$1 \le d_1 + d_1 + d_{n-2} \le 2,$$

$d_{n-2} \neq d_1$, i.e., $d_{n-1} = d_{n-2}$. Again by (2), we must have $d_2 = d_1$. Since

$$1 \le d_1 + d_2 + d_{n-3} \le 2,$$

we have $d_{n-3} = d_{n-2} = d_{n-1}$. By repeating the above argument, we obtain the series of equalities:

$$d_1 = d_2 = d_3 = \ldots = d_{\lfloor \frac{n}{2} \rfloor},$$

$$d_{n-1} = d_{n-2} = \ldots = d_{\lceil \frac{n}{2} \rceil}.$$

We note that n must be odd, otherwise $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil$. However, in this case,

$$d_1 + d_{\lfloor \frac{n}{2} \rfloor} + d_{\lfloor \frac{n}{2} \rfloor} = 0 \text{ or } 3,$$

a contradiction. ∎

Maskin (1995) also provides a converse of sorts to the above.[5] Let g be any rule that associates with each $\mathbf{P} \in \Omega^n$ a pairwise ordering of the alternatives in A. There is no requirement that g produce a transitive ordering of the elements of A, i.e., g need not be a ASWF on every domain Ω.

[5] See Campbell and Kelly (2000) for the same result derived under slightly weaker conditions.

Theorem 7 *Suppose that g is anonymous, neutral, satisfies unanimity and is not majority rule. Then there exists a domain Ω on which g is a not an ASWF but majority rule is.*

Proof
Anonymity implies that g is not dictatorial. Thus, there has to be a domain Ω and some profile $\mathbf{P} \in \Omega^n$ on which g generates an intransitive order. Since g satisfies independence of irrelevant alternatives, it can be described by 'decisiveness' variables. Call them d'.

Given that g is not a ASWF on Ω, there is a triple of alternatives $x, y, z \in A$ and an appropriate partition of the agents such that constraint (3) is violated. Suppose first that Ω does not admit a Condorcet triple in the alternatives x, y, z. Let Π be the set of orderings of x, y, z admissible under Ω. Fix an ordering σ of elements of $A \setminus \{x, y, z\}$. Let Ω' be the set of all preference orderings of the form (π, σ) where $\pi \in \Pi$. It is easy to see that majority rule is a ASWF on this domain but g cannot be since it would violate (3) with respect to $\{x, y, z\}$. Hence we may assume that every violation of (3) by d' on any domain is associated with a Condorcet triple.

Let a and b be two positive integers. We claim that $d'_a = d'_b = 1 \;\Rightarrow a + b > n$. To see why, suppose not and consider the domain consisting of the following three orderings: $\{xyz, yzx, yxz\}$. This domain does not admit a Condorcet triple and so g, equivalently d' defines an ASWF on it. Suppose now a profile where a agents have the ranking xyz, b agents have the ranking yzx and the remaining $n - a - b$ agents have the ranking yxz. The first set of a agents are the only ones to rank x above y. Since $d'_a = 1$, on this domain g ranks x above y. A similar argument applies to the second set of b agents and the ordered pair zx. However, unanimity requires that g rank y above z. Hence g is not an ASWF on this domain, a contradiction.

Next we claim that if $a \leq b$ then $d'_a = 1 \;\Rightarrow\; d'_b = 1$. Suppose not. By (2) it follows that $d'_{n-b} = 1$. From the previous claim, $d'_a = d'_{n-b} = 1$ implies that $a + n - b > n$ which cannot be since $a \leq b$.

Let r be the smallest integer such that $d'_r = 1$. Suppose first that $r < n/2$. Now $d'_r = 1$ implies $d'_{r+1} = 1$. But $r + (r+1) \leq n$ a contradiction. Now assume that $r > n/2$. If $d'_{\frac{(n+1)}{2}} = 0$ we have from (2) that $d'_{\frac{(n-1)}{2}} = 1$ which contradicts the choice of r. Since $d'_{\frac{(n+1)}{2}} = 1$ it follows that $d'_a = 1$ for all $a \geq (n+1)/2$. That is g is majority rule, contradiction. ■

7 Social Choice Functions

A **Social Choice Function** maps profiles of preferences into a single alternative. The analog of Arrow's impossibility theorem for social choice functions is the Muller-Satterthwaite theorem (1977). The counterpart of Unanimity and the Independence of Irrelevant Alternatives condition for Social Choice Functions are called **pareto optimality** and **monotonicity**. To define them, denote the preference ordering of agent i in profile $\mathbf{P}$ by $\mathbf{p_i}$.

1. **Pareto Optimality:** Let $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}y$ for all $\mathbf{p} \in \mathbf{P}$. Then $f(\mathbf{P}) \neq y$.

2. **Monotonicity:** For all $x \in A$, $\mathbf{P}, \mathbf{Q} \in \Omega^n$ if $x = f(\mathbf{P})$ and $\{y : x\mathbf{p_i}y\} \subseteq \{y : x\mathbf{q_i}y\}\ \forall i$ then $x = f(\mathbf{Q})$.

We call a Social Choice Function that satisfies pareto-optimality and monotonicity an Arrovian social choice function (ASCF). The theorem of Muller and Satterthwaite says that when $\Omega = \Sigma$ all ASCF's are dictatorial.[6] Here we will prove something stronger using the methods introduced earlier.

For each subset S of agents and ordered pair of alternatives (x, y), denote by $[S, x, y]$ the set of all profiles where agents in S rank x first and y second, and agents in S^c rank y first and x second. By the hypothesis on Ω this collection is well defined.

For any profile $\mathbf{P} \in [S, x, y]$ it follows by pareto optimality that $f(\mathbf{P}) \in \{x, y\}$. By monotonicity, if $f(\mathbf{P}) = x$ for one such profile $\mathbf{P}$ then $f(\mathbf{P}) = x$ for all $\mathbf{P} \in [S, x, y]$.

Suppose then for all $\mathbf{P} \in [S, x, y]$ we have $f(\mathbf{P}) \neq y$. Let $\mathbf{Q}$ be any profile where all agents in S rank x above y, and all agents in S^c rank y above x. We show next that $f(\mathbf{Q}) \neq y$ too.

Suppose not. That is $f(\mathbf{Q}) = y$. Let $\mathbf{Q}'$ be a profile obtained by moving x and y to the top in every agents ordering but preserving their relative position within each ordering. So, if x was above y in the ordering under $\mathbf{Q}$, it remains so under $\mathbf{Q}'$. Similarly if y was above x.

By monotonicity $f(\mathbf{Q}') = y$. But monotonicity with respect to $\mathbf{Q}'$ and $\mathbf{P} \in [S, x, y]$ implies that $f(\mathbf{P}) = y$ a contradiction.

Hence, if there is one profile in which all agents in S rank x above y, and all agents in S^c rank y above x, and y is not selected, then all profiles with such a property will not select y. This observation allows us to describe ASCF's using the following variables.

For each $(x, y) \in A^2$ a 0-1 variable as follows:

- $g_S(x, y) = 1$ if when all agents in S rank x above y and all agents in S^c rank y above x then y is never selected,

- $g_S(x, y) = 0$ otherwise.

If E is the set of all candidates we set $g_E(x, y) = 1$ for all $(x, y) \in A^2$. This ensures pareto optimality.

Consider a $P \in \Omega^n$, $(x, y) \in A^2$ and subset S of agents such that all agents in S prefer x to y and all agents in S^c, the complement of S, prefer y to x. Then, $g_S(x, y) = 0$ implies that $g_{S^c}(y, x) = 1$ to ensure a selection. Hence for all S and $(x, y) \in A^2$ we have

$$g_S(x, y) + g_{S^c}(y, x) = 1. \tag{23}$$

First we mimic the derivation of inequalities (4) and (5). Suppose there are $\mathbf{p}, \mathbf{q} \in \Omega$ and three alternatives x, y and z such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Fix a subset S of agents and

[6] The more well known result about strategy proof social choice functions is due to Gibbard (1973) and Satterthwaite (1975). It is a consequence of Muller-Satterthwaite (1977).

consider a profile where all agents in S have $\mathbf{p}$ and all agents in S^c have $\mathbf{q}$. Suppose we set $g_S(x,y) = 1$. For the chosen profile this means y will not be selected. From pareto optimality we know that z cannot be selected. Hence $g_S(x,z) = 1$ to be consistent. Summarizing:

$$g_S(x,y) = 1 \Rightarrow g_S(x,z) = 1$$

and

$$g_S(z,x) = 1 \Rightarrow g_S(y,x) = 1.$$

These two conditions can be formulated as inequalities:

$$g_S(x,y) \leq g_S(x,z), \text{ and} \tag{24}$$

$$g_S(z,x) \leq g_S(y,x). \tag{25}$$

Note that these inequalities hold only if there exists $\mathbf{p}, \mathbf{q} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$.

Using these inequalities, for each set S, we can define a directed graph D^Ω_S where the nodes in the graph correspond to $g_S(x,y)$ and there is a directed arc from the node $g_S(x,y)$ to $g_S(x,z)$ if (24) holds. Similarly, we have an arc from $g_S(z,x)$ to $g_S(y,x)$. Note that the graph D^Ω_S does not depend on the choice of S, hence in the rest of this section, we drop the subscript S and refer to the graph as D^Ω.

Theorem 8 *Let Ω be such that for all $(x,y) \in A^2$ there is a $\mathbf{p} \in \Omega$ such that $r^1(\mathbf{p}) = x$ and $r^2(\mathbf{p}) = y$. Then, all the ASCF's are dictatorial.*

Proof: Suppose there exist an ASCF that is not dictatorial. The corresponding g_S variables are thus well defined. We first prove, using a combination of pareto optimality and monotonicity, that

$$g_S(x,y) \leq g_T(x,y) \ \forall S \subset T. \tag{26}$$

To see why it must hold, suppose not. Then we can find a specification of the g variables that satisfies all the other inequalities but $g_S(x,y) = 1$ and $g_T(x,y) = 0$ for some $(x,y) \in A^2$.

Let $\mathbf{P}$ be a profile where all agents in S rank x first, y second and all agents in S^c rank y first and x second. On such a profile, g requires that y not be selected. Pareto optimality would dictate that x is the outcome.

Let $\mathbf{Q}$ be a profile where all agents in T rank x first and y second , all agents in T^c rank y first and x second. Since $g_T(x,y) = 0$ it follows that $g_{T^c}(y,x) = 1$. Thus, on this profile g would eliminate x. Pareto optimality would require that y be the outcome. But this violates monotonicity with respect to $\mathbf{P}$ and $\mathbf{Q}$.

¿From the assumption that D^Ω is strongly connected we know that every integral solution to the above collection of inequalities must have for each $S \subset E$, $g_S(x,y) = 1 \ \forall (x,y) \in A^2$ or $g_S(x,y) = 0 \ \forall (x,y) \in A^2$. If there is a $j \in E$ such that $g_S(x,y) = 1 \ \forall (x,y) \in A^2$, and all $S \ni j$ we are done. Suppose not. Pick any triple $\{x,y,z\} \in A$. From amongst all $S \subseteq E$ with $g_S(a,b) = 1 \ \forall (a,b) \in A^2$ choose one that is minimal (i.e. S has the smallest cardinality).

Suppose $|S| = 1$. From (26) it follows that $g_T(x, y) = 1$ for all T that contain S and we are done.

If $|S| > 1$, Pick any proper subset of agents $T \subset S$ and consider a profile $\mathbf{P}$ where agents in T rank x first, y second and z anywhere below y. Every agent in $S \setminus T$ ranks z first, x second, and y anywhere below x. Lastly, all agents in S^c rank y first, z second and x anywhere below z. Such profiles exist by assumption.

Since $g_S(x, y) = 1$ for this profile we know that y can not be selected. By minimality of S, we have that $g_T(x, z) = 0 \; \forall (x, y) \in A^2$. From (23) we get that $g_{E\setminus T}(z, x) = 1$. For the profile under consideration all agents in $E \setminus T$ rank z above x. Hence x cannot be selected. Similarly $g_{S\setminus T}(z, y) = 0$. Hence $g_{S^c \cup T}(y, z) = 1$. Since all agents in $S^c \cup T$ rank y above z, it follows that for the profile under consideration that z is never selected. Hence no alternatives from the set $\{x, y, z\}$ are selected.

We complete the argument by showing that for this profile no $a \in A \setminus \{x, y, z\}$ can be selected either. Suppose not.

If agents in T rank a below z, then all agents in this profile rank z above a and so by pareto optimality alternative a can never be selected. Thus, in this profile, a must be ranked above z and below y by the agents in S. A similar argument shows that agents in S^c rank y above a above x, and agents in $S \setminus T$ rank x above a above y. Hence Ω admits the following two orderings: xya and yax. Since $g_S(x, y) = 1$ and D^Ω is strongly connected, it follows that $g_S(x, a) = 1$. But this means that a cannot be selected, a contradiction. ∎

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