Cut generation for mixed 0-1 quadratically constrained programs

G. Iyengar[1] M. T. Çezik

1 Introduction

The goal of this research is to develop cut generation methods for mixed 0-1 second-order cone programs (SOCPs) having the following form,

$$\begin{array}{lll} \text{minimize} & f^T x, & \\ \text{subject to} & \|A_i x + b_i\| \leq c_i^T x + d_i, & i = 1, \ldots, N, \\ & x_k \in \{0, 1\}, & k = 1, \ldots, p, \end{array} \tag{1}$$

where $x \in \mathbf{R}^n$ is the optimization variable, and the problem parameters are $f \in \mathbf{R}^n$, $A_i \in \mathbf{R}^{(n_i-1)\times n}$, $b_i \in \mathbf{R}^{n_i-1}$, $c_i \in \mathbf{R}^n$ and $d_i \in \mathbf{R}$. The norm appearing in the constraints is the standard Euclidean norm, i.e. $\|u\| = \sqrt{u^T u}$. Second-order cone programming is a very flexible modeling tool and has found many applications. For a detailed survey of the various applications of SOCPs see [8]. In many of these applications binary decision variables naturally emerge.

In this abstract we discuss cut generation methods for a restricted class of SOCPs where $c_i \equiv 0$. An SOCP of this class can be reformulated as the following mixed 0-1 convex quadratically constrained program (CQCP),

$$\begin{array}{lll} \text{minimize} & f^T x, & \\ \text{subject to} & x^T Q_i x + 2q_i^T x + \gamma_i \leq 0, & i = 1, \ldots, N, \\ & x_k \in \{0, 1\}, & k = 1, \ldots, p, \end{array} \tag{2}$$

where $Q_i = A_i^T A_i \succeq 0$, $q_i = A_i^T b_i$ and $\gamma_i = b_i^T b_i - d_i^2$.

Mixed 0-1 CQCPs are a special case of the quadratically constrained quadratic programs (QCQP) where all the non-convexity comes from the binary constraints. For a discussion of applications of QCQPs see [11, 3]. Recently, Kim and Kojima [6] have developed CQCP-based relaxation for non-convex quadratic optimization problems. The cut generation methods developed here can be used to sequentially tighten this relaxation.

1.1 Previous work and outline of main results

The mixed 0-1 CQCP is solved by the usual two step procedure. In the first step, the binary constraints $x_k \in \{0, 1\}$ are relaxed to the convex quadratic constraint $x_k(x_k - 1) \leq 0$ and the resulting CQCP is solved to optimality. If the solution $\bar{x}$ is feasible for the mixed 0-1 CQCP (2) then $\bar{x}$ is the optimal solution. Otherwise, one generates a convex constraint

[1] G. Iyengar, IEOR Department, Columbia University, 331 S.W. Mudd Bldg., 500 W 120th Street, NY, NY 10027-6699, USA. Email: garud@ieor.columbia.edu

that is valid for the feasible set of the mixed 0-1 problem (2) but is violated by the current iterate $\bar{x}$, i.e. a convex constraint that "cuts" the current iterate. This convex constraint is added to the convex relaxation and the steps repeated.

This method has its origins in solving integer programming formulations of combinatorial optimization problems. In this setting, the convex relaxation is an LP and linear constraints are used to "cut" the iterates – hence the name cutting plane method. The cutting planes are generated using specific problem structure and, therefore, the method does not generalize to mixed 0-1 LPs that are not combinatorial in nature. Balas, Ceria and Cornuèjols [2] developed a disjunctive programming based cutting plane algorithm for general mixed 0-1 LPs that does not rely on the problem structure.

A brief description of the disjunctive cut generation method is as follows. Let K denote the feasible set of the LP relaxation of the feasible set K° of an integer program. Suppose the j-th component of the current iterate $\bar{x}_j \notin \{0,1\}$, i.e. $0 < \bar{x}_j < 1$, for some $j \leq p$. Then $\bar{x} \notin K_j$ where K_j is defined as

$$K_j = \mathbf{conv}\left((K \cap \{x \mid x_j = 0\}) \cup (K \cap \{x \mid x_j = 1\})\right),$$

i.e. K_j is the convex hull of the disjunctive constraints $\{x \in K \mid x_j \in \{0,1\}\}$.

Since $\bar{x} \notin K_j$ there exists a linear inequality, $q^T x + \gamma \leq 0$, that is valid for all $x \in K_j$, and *a fortiori* for $x \in \mathbf{conv}(K^\circ) \subseteq K_j$, but is violated by $\bar{x}$. To be successful, a cutting plane method must be able to efficiently generate this violated inequality. Using disjunctive programming [1], Balas et al showed that the set of linear inequalities valid for the set K_j is a well-defined polyhedral cone and, hence, cut generation amounts to solving an LP [2]. This approach can also be interpreted as lift-and-project cut generation [12, 9] where the lifting is done only along the variable x_j.

Stubbs and Mehrotra generalized this method to mixed 0-1 convex programs [14]. However, in this general setting, cut generation is not efficient and, as a consequence, the method does not yield an efficient optimization procedure. We have recently extended the disjunctive programming-based approach to mixed 0-1 semidefinite programming (SDP) and shown that cuts can be efficiently generated by solving an SDP [5, 4]. Since mixed 0-1 CQCPs can be reformulated as mixed 0-1 SDPs, our method works for mixed 0-1 CQCPs as well. However, the complexity of the method is prohibitive because the SDPs corresponding to CQCPs are typically very large.

In a related development, Kojima and Tunçel recently developed a SDP-based sequential convexification procedure for solving nonconvex quadratic optimization problems [7]. In this work, we combine this convexification procedure with disjunctive programming to generate cuts for mixed 0-1 QCQPs. In addition to generating linear cuts, we present methods for generating convex quadratic cuts, i.e. convex quadratic constraints valid for K° but violated by the current iterate $\bar{x}$. Convex quadratic cuts were introduced by Stubbs and Mehrotra in the context of general mixed 0-1 convex programming [13, 15]. Their results when restricted to the special case of CQCPs are weaker than those presented here.

There are several motivations for studying convex quadratic cuts. Although it is clear that the "deepest" cut separating the iterate $\bar{x}$ from the set K_j will be linear, convex quadratic cuts may provide a better localization for $\mathbf{conv}\,(K^\circ)$. Unlike in the case of mixed 0-1 LPs, linear cuts for mixed 0-1 QCQPs are not likely to be facet defining. There is, therefore, the possibility that quadratic cuts may provide a tighter description of $\mathbf{conv}\,(K^\circ)$. Preliminary experiments suggest that the performance of the quadratic cuts is relatively insensitive to perturbations in the objective f. Thus, quadratic cuts could possibly be re-used with several different objectives. Convex quadratic cuts can be used in the context of the CQCP relaxations developed by Kim and Kojima [6] to sequentially tighten the relaxation. And finally, convex quadratic cuts are, in a sense, natural for mixed 0-1 CQCP; therefore, their properties are worth studying.

In §2, we present the basic results for generating linear and quadratic cuts via semidefinite lifting and disjunctive programming. In this section, the cut generation step involves solving an SDP. In §3 we show that all linear cuts and a restricted class of quadratic cuts can be generated using SOCPs. This is important because the complexity of SOCPs is at least an order of magnitude lower than that of SDPs.

In §4 we introduce the "volume" of a convex quadratic cut as a measure of the localization it provides. The intuition is that if $\mathbf{int}(\mathbf{conv}\,(K^\circ))$ is not empty, then a cut with lower volume will provide a better localization. However, such a cut may be very "shallow" in terms of separating the current iterate. To balance these opposing objectives, we have developed a cut generation method that trades off the "depth" of the cut with its "volume". In §5 we discuss some extensions to mixed 0-1 SOCPs.

We are still the in the process of validating the computational performance of the methods proposed here.

2 Cuts via semidefinite programming

Denote the feasible region of the mixed 0-1 CQCP (2) by K°, i.e.

$$K^\circ = \left\{ x \,\middle|\, x^T Q_i x + 2q_i^T x + \gamma_i \leq 0, i = 1, \ldots, N, x_k \in \{0,1\}, k = 1, \ldots, p \right\}, \tag{3}$$

Define a convex relaxation K of the set K° as follows,

$$K = \left\{ x \,\middle|\, x^T Q_i x + 2q_i^T x + \gamma_i \leq 0, i = 1, \ldots, N' \right\}, \tag{4}$$

where the constraints $x_i \in [0,1]$ are imposed by the convex quadratic inequality $x_i(x_i - 1) \leq 0$. The semidefinite lifting $\widehat{K}$ of the set K is defined as follows,

$$\widehat{K} \;=\; \left\{ (x, X) \in \mathbf{R}^n \times \mathcal{S}^n \,\middle|\, X - xx^T \succeq 0, Q_i \bullet X + 2q_i^T x + \gamma_i \leq 0, \forall i \right\}, \tag{5}$$

where $\mathcal{S}^n$ denotes the space of all symmetric $n \times n$ matrices, $A \bullet B = \mathbf{Tr}(AB)$ is the usual inner product on matrices and $A \succeq 0$ denotes that the matrix A is positive semidefinite.

Let $\mathcal{P}\widehat{K}$ denote the projection of the set $\widehat{K}$ on the x-subspace, i.e.

$$\mathcal{P}\widehat{K} = \left\{ x \,\middle|\, \exists X \in \mathcal{S}^n \text{ such that } \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0, \begin{bmatrix} \gamma_i & q_i^T \\ q_i & Q_i \end{bmatrix} \bullet \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \leq 0, \forall i \right\}. \qquad (6)$$

This projected set $\mathcal{P}\widehat{K}$ is called the semidefinite lift-and-project relaxation of the set K. Since $x \in K$ implies that $(x, xx^T) \in \widehat{K}$, it follows that $\mathcal{P}\widehat{K} \supseteq K$. Kojima and Tunçel recently showed that if $Q_i \succeq 0$, then the semidefinite lift-and-project relaxation is, in fact, the same as the original set, i.e. $\mathcal{P}\widehat{K} = K$ (see Theorem 4.2 in [7]). Thus, (6) reformulates the quadratically constrained set K as a linearly constrained set in the space of the lifted variables (x, X). This linear description is the basis of linear and quadratic cut generation.

If the current iterate $\bar{x}$ is an extreme point and $0 < \bar{x}_j < 1$ for some $j \leq p$, then $\bar{x} \notin K_j$, where

$$K_j = \mathbf{conv}\left(K \cap \{x \mid x_j \in \{0,1\}\}\right) \qquad (7)$$

Since $K = \mathcal{P}\widehat{K}$, it follows that $K_j = \mathcal{P}\widehat{K}_j$, where $\widehat{K}_j$ is analogously defined as follows,

$$\widehat{K}_j = \mathbf{conv}\left(\widehat{K} \cap \{x \mid x_j \in \{0,1\}\}\right).$$

We generate linear and convex quadratic cuts separating $\bar{x}$ from K_j ($\supseteq K^\circ$) by projecting down linear inequalities of the lifted set $\widehat{K}_j$. Thus, in order to generate the cuts we need to characterize the set $\widehat{K}_j^*$ of all valid linear inequalities for the lifted set $\widehat{K}_j$. By suitably extending Balas' results [1] on disjunctive programming, we prove the following result.

Lemma 1 *Let* $K = \{x \in \mathbf{R}^n \mid x^T Q_i x + 2q_i^T x + \gamma_i \leq 0, 1 \leq i \leq N', Q_i \succeq 0, \ \forall i\}$. *Then* $(x, X) \in \hat{\mathcal{K}}_j$ *if and only if there exist,* (y, Y), (z, Z) *such that* $x = y + z$, $X = Y + Z$ *and*

$$\begin{bmatrix} y_0 & y^T \\ y & Y \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} z_0 & z^T \\ z & Z \end{bmatrix} \succeq 0,$$

$$Q_i \bullet Y + 2q_i^T y + \gamma_i y_0 \leq 0, \quad i = 1, \ldots, N',$$

$$Q_i \bullet Z + 2q_i^T z + \gamma_i z_0 \leq 0, \quad i = 1, \ldots, N',$$

$$y_i = 0, \quad z_i = z_0, \quad y_0 + z_0 = 1.$$

Using semidefinite programming duality, we characterize the polar set $\widehat{K}_j^*$ as follows.

Lemma 2 *If* $\mathbf{int}(\mathbf{conv}\,(K^\circ)) \neq \emptyset$*, then the inequality* $Q \bullet X + 2q^T x + \gamma \leq 0$ *is valid for* K_j*, i.e.* $(Q, q, \gamma) \in \widehat{K}_j^*$*, if and only if there exist* $u_0, v_0 \in \mathbf{R}$*,* $u, v \in \mathbf{R}^n$*,* $u \geq 0, v \geq 0$ *such that*

$$M_u = \begin{bmatrix} \sum_{i=1}^{N'} u_i\gamma_i - \gamma & (\sum_{i=1}^{N'} u_i q_i + u_o e_j - q)^T \\ \sum_{i=1}^{N'} u_i q_i + u_o e_j - q & \sum_{i=1}^{N'} u_i Q_i - Q \end{bmatrix} \succeq 0,$$

$$M_v = \begin{bmatrix} \sum_{i=1}^{N'} v_i\gamma_i - 2v_0 - \gamma & (\sum_{i=1}^{N'} v_i q_i + u_o e_j - q)^T \\ \sum_{i=1}^{N'} v_i q_i + u_o e_j - q & \sum_{i=1}^{N'} v_i Q_i - Q \end{bmatrix} \succeq 0.$$

where e_j *is the* j*-th basis vector.*

The Slater-type condition, $\mathbf{int}(\mathbf{conv}\,(K^\circ)) \neq \emptyset$, is only needed to ensure strong duality and the result remains true under any other condition that ensures strong duality.

2.1 Linear cuts

Suppose the current iterate $\bar{x} \notin K_j$. Then, by minimum norm duality, the distance of $\bar{x}$ from K_j is given by the solution of the following dual problem,

$$\begin{array}{ll} \text{maximize} & q^T\bar{x} + \gamma, \\ \text{subject to} & q^T x + \gamma \leq 0, \quad \forall x \in K_j \\ & \|q\| \leq 1. \end{array} \tag{8}$$

Let (q^*, γ^*) be the optimal solution of this program. Since $\bar{x} \notin K_j$, it must be that $(q^*)^T\,\bar{x} + \gamma^* > 0$, i.e. the linear inequality $(q^*)^T x + \gamma^* \leq 0$ separates $\bar{x}$ from $K_j \supseteq K^\circ$.

Since $K_j = \mathcal{P}\widehat{K}_j$, it follows that $q^T x + \gamma \leq 0$ for all $x \in K_j$ if and only if $(0, \frac{1}{2}q, \gamma) \in \widehat{K}_j^*$. The norm constraint $\|q\| \leq 1$ can be reformulated as the semidefinite constraint,

$$\begin{bmatrix} 1 & q^T \\ q & I \end{bmatrix} \succeq 0,$$

Thus, from Lemma 2, it follows that the cut generation problem (8) is an SDP.

2.2 Quadratic cuts

Suppose the current iterate $\bar{x} \notin K_j$. Then for any matrix $\bar{X} \succeq \bar{x}\bar{x}^T$, the pair $(\bar{x}, \bar{X}) \notin \widehat{K}_j$. In particular $(\bar{x}, \bar{x}\bar{x}^T) \notin \widehat{K}_j$. The separation of the point $(\bar{x}, \bar{x}\bar{x}^T)$ from the set $\widehat{K}_j$ is given by the solution of the dual problem,

$$\begin{array}{ll} \text{maximize} & Q \bullet (\bar{x}\bar{x}^T) + 2q^T\bar{x} + \gamma \\ \text{subject to} & (Q, q, \gamma) \in \widehat{K}_j^*, \\ & \|Q\| + 2\,\|q\| \leq 1. \end{array} \tag{9}$$

The optimal solution of this problem is defines the "deepest" cut separating $(\bar{x}, \bar{x}\bar{x}^T)$ from $\widehat{K}_j$. Lemma 2 establishes that this cut generation problem is an SDP.

Next, we show that if $Q \succeq 0$ and $Q \bullet X + 2q^T x + \gamma \leq 0$ is a valid linear inequality for $\widehat{K}_j$, then its projection on the x-subspace is a valid convex quadratic inequality for the set K_j.

Lemma 3 *Let $Q \bullet X + 2q^T x + \gamma \leq 0$ be a valid linear inequality for $\widehat{K}_j$ with $Q \succeq 0$ and $Q \bullet (\bar{x}\bar{x}^T) + 2q^T\bar{x} + \gamma > 0$. Then, $x^TQx + 2q^Tx + \gamma \leq 0$ is valid convex quadratic inequality for K_j and separates it from $\bar{x}$, i.e. (Q, q, γ) induces a valid convex quadratic cut.*

The result follows from the fact that if $Q \succeq 0$ and $(x, X) \in \widehat{K}_j$ then $Q \bullet X \geq Q \bullet (xx^T) = x^TQx$. As the last step, we establish that the cut generation problem (9) with Q restricted to be positive semidefinite separates the current iterate $\bar{x}$ from K_j.

Lemma 4 *If $\mathbf{int}(\mathbf{conv}\,(K^\circ)) \neq \emptyset$ and $\bar{x} \notin K_j$, then there exists $(Q, q, \gamma) \in \widehat{K}_j^*$ with $Q \succeq 0$ such that $\bar{x}^TQ\bar{x} + 2q^T\bar{x} + \gamma > 0$.*

Proof: The dual of (9) with the extra constraint $Q \succeq 0$ is given by

$$\begin{array}{rl} \text{minimize} & 2\left\|x - \bar{x}\right\| + \left\|X - \bar{x}\bar{x}^T - V\right\|, \\ \text{subject to} & (x, X) \in \widehat{K}_j, \\ & V \succeq 0. \end{array} \tag{10}$$

Since $\bar{x} \notin K_j = \mathcal{P}\widehat{K}_j$, it follows that the optimal value of (10) is strictly positive, i.e. the restricted dual separates the point $(\bar{x}, \bar{x}\bar{x}^T)$. ■

Thus, we have a procedure for generating convex quadratic cuts for K_j. From Lemma 2 and Lemma 4, it follows that quadratic cut generation problem is an SDP.

3 Cuts via second-order cone programming

From Lemma 2, it follows that $2q^Tx + \gamma \leq 0$ is a valid linear inequality for K_j provided there exist multipliers $u, v \geq 0$ and $u_0, v_0 \in \mathbf{R}$ such that $M_u \succeq 0$ and $M_v \succeq 0$ with $Q \equiv 0$. If $\sum_{i=1}^{N'} u_iQ_i \succ 0$ then, by taking Schur complements, it follows that $M_u \succeq 0$ if and only if

$$\sum_i u_i\gamma_i - \gamma \geq \Big(\sum_i u_iq_i + u_oe_j - q\Big)^T\Big(\sum_i u_iQ_i\Big)^{-1}\Big(\sum_i u_iq_i + u_oe_j - q\Big) \tag{11}$$

This is a matrix-fractional constraint in the variables (q, γ, u, u_0) (see [10, 8] for details).

Let $Q_i = L_i^TL_i$ where $L_i \in \mathbf{R}^{\mathbf{rank}(Q_i)\times n}$ (if the CQCP came from an SOCP (1) then $L_i = A_i$). From the results in [10, 8], it follows that (u, u_0) satisfies (11) if and only if there exists $w_i \in \mathbf{R}^{\mathbf{rank}(Q_i)}$ and $t \in \mathbf{R}^N$ such that

$$\begin{array}{rcll} \sum_i L_iw_i & = & \sum_i u_iq_i + u_oe_j - q, & \\ w_i^Tw_i & \leq & t_iu_i, & \text{for all } i, \\ \mathbf{1}^Tt & \leq & \sum_i u_i\gamma_i - \gamma. & \end{array} \tag{12}$$

The constraint $w_i^Tw_i \leq t_iu_i$ can be rewritten as the following second-order cone constraint,

$$\left\|\begin{bmatrix} 2w_i \\ (u_i - t_i) \end{bmatrix}\right\| \leq u_i + t_i.$$

Thus, the constraint $M_u \succeq 0$ is equivalent to the following set of second-order cone constraints,

$$\begin{array}{rcll} \sum_i L_i w_i & = & \sum_i u_i q_i + u_0 e_j - q, & \\ \left\| \begin{bmatrix} 2w_i \\ u_i - t_i \end{bmatrix} \right\| & \leq & u_i + t_i, & \text{for all } i, \\ \mathbf{1}^T t & \leq & \sum_i u_i \gamma_i - \gamma, & \\ u & \geq & 0. & \end{array} \tag{13}$$

Similarly, the constraint $M_v \succeq 0$ can be reformulated as a second-order cone constraint. Therefore, linear cut generation reduces to solving an SOCP.

This SOCP-based method can be extended to quadratic cut generation provided the semidefinite constraint $\sum_i u_i Q_i \succeq Q$ can be imposed implicitly.

Let $Q_i = L_i^T \Lambda_i L_i$ be the spectral decomposition of the matrix Q_i and let $\lambda_i = \mathbf{diag}(\Lambda_i)$ be the vector of diagonal elements. Suppose the matrix Q defining the cut is restricted to be of the following form,

$$Q = \sum_{i=1}^{N} L_i^T \Sigma_i L_i, \tag{14}$$

where Σ_i is a diagonal matrix and the diagonal $\sigma_i = \mathbf{diag}(\Sigma_i)$ satisfies, $\sigma_i \leq u_i \lambda_i$.

Then $\sum_i u_i Q_i - Q = \sum_i L_i^T (u_i \Lambda_i - \Sigma_i) L_i \succeq 0$. Similarly, the linear constraint $\sigma_i \leq v_i \lambda_i$ ensures that $\sum_i v_i Q_i - Q \succeq 0$. Thus, the separation problem over this restricted class of cuts is an SOCP. Since the linear cuts cut the current iterate $\bar{x}$, it follows that this restricted class of quadratic cuts will also separate the iterate $\bar{x}$ from K_j.

4 Minimum volume cut generation

Let $F = \{x \mid x^T Q x + 2q^T x + \gamma \leq 0\}$ denote the feasible set of any convex quadratic inequality $x^T Q x + 2q^T x + \gamma \leq 0$ valid for K_j and let $\mathbf{vol}(F)$ denote the volume of the set F. If $\mathbf{int}(\mathbf{conv}\,(K^\circ)) \neq \emptyset$, i.e. the convex hull is full dimensional, then $\mathbf{vol}(F)$ is a measure of the localization provided by the cut. In this section, we will develop a cutting plane method that trades off the volume and the depth of a convex quadratic cut.

Since $\mathbf{rank}(Q) < n$ implies that the volume $\mathbf{vol}(F)$ is infinite, we restrict ourselves to $Q \succ 0$. If $Q \succ 0$ then $\mathbf{vol}(F)$ is given by,

$$\mathbf{vol}(F) = \mathbf{vol}\left(\left\{ x \,\middle|\, \left\| Q^{\frac{1}{2}} x + Q^{-\frac{1}{2}} q \right\| \leq \sqrt{\left\| Q^{-\frac{1}{2}} q \right\|^2 - \gamma} \right\}\right) = C\,\mathbf{det}\left(Q^{-\frac{1}{2}}\right) \left(\left\| Q^{-\frac{1}{2}} q \right\|^2 - \gamma \right)^{\frac{n}{2}},$$

where C is a dimension dependent constant. Define the function $\mu(Q, q, \gamma)$ as follows,

$$\begin{aligned} \mu(Q, q, \gamma) &= -\log(\mathbf{vol}(F)) + M \log(\bar{x}^T Q \bar{x} + 2q^T \bar{x} + \gamma), \\ &= \frac{1}{2} \log\det(Q) - \frac{n}{2} \log\left(q^T Q^{-1} q - \gamma\right) + M \log(\bar{x}^T Q \bar{x} + 2q^T \bar{x} + q), \end{aligned} \tag{15}$$

where M is a constant. Since the deepest cut is likely to be linear (i.e. infinite volume), the function μ trades off the two opposing objectives of maximal separation and minimal volume. The optimal trade-off would be given by the solution of the quadratic cut generation SDP (9) with the function μ as the objective. Unfortunately μ is not convex on the feasible set of (9) and, therefore, the optimization problem cannot be solved efficiently.

However, if one replaces, $\|Q\| + 2\|q\| \leq 1$, by the following normalization condition,

$$q^T Q^{-1} q - \gamma \leq 1 \Leftrightarrow \begin{bmatrix} 1+\gamma & q^T \\ q & Q \end{bmatrix} \succeq 0,$$

then this problem becomes convex. With this normalization the optimal solution is an extreme point of the dual set K_j^* but the cut violation does not have a minimum norm interpretation. The following Lemma summarizes the main result of this section.

Lemma 5 *Let (Q^*, q^*, γ^*) be the optimal solution of the following optimization problem,*

$$\begin{array}{ll} \textit{maximize} & \frac{1}{2}\log\mathbf{det}(Q) + M\log(\bar{x}^T Q\bar{x} + 2q^T\bar{x} + \gamma), \\ \textit{such that} & (Q, q, \gamma) \in \widehat{K}_j^*, \\ & q^T Q^{-1} q \leq 1 + \gamma. \end{array} \tag{16}$$

then the objective value at (Q^*, q^*, γ^*) is equal to $\mu(Q^*, q^*, \gamma^*)$, and (Q^*, q^*, γ^*) induces a valid convex quadratic cut for K_j.

The optimization problem (16) belongs to a class of problems called determinant maximization problems. Vandenberghe et al have developed very efficient interior point methods for solving these problems [16].

5 Cuts generation for mixed 0-1 SOCP

In this section, we present limited extensions of the results for mixed 0-1 CQCPs to general SOCPs. As before, let

$$K^\circ = \{x \mid \|A_i x + b_i\| \leq c_i^T x + d_i, x_k \in \{0,1\}, k = 1, \ldots, p\}$$

denote the feasible set of the mixed 0-1 SOCP and

$$K = \{x \mid \|A_i x + b_i\| \leq c_i^T x + d_i, x_k \in [0,1], k = 1, \ldots, p\},$$

denote the convex relaxation and $K_j = \mathbf{conv}\left(\{x \in K \mid x_j \in \{0,1\}\}\right)$

5.1 Linear cuts

Extending the standard results on disjunctive programming [1], we show that $x \in K_j$ if and only if there exist $(y, y_0), (z, z_0) \in \mathbf{R}^{n+1}$ such that

$$\begin{array}{lll} \|A_i y + b_i y_0\| \leq c_i^T y + d_i y_0, & \|A_i z + b_i z_0\| \leq c_i^T z + d_i z_0, & i = 1, \ldots, N, \\ y_0 + z_0 = 1, \quad y_j = y_0, \quad z_j = 0. & & \end{array}$$

From this formulation it follows that the multipliers (u, u_0) and (v, v_0) in the dual representation of the set K_j satisfy the second-order cone constraints (12) in §3. Hence, the SOCP-based linear cut generation method for mixed 0-1 CQCPs extends to mixed 0-1 SOCPs.

5.2 Quadratic cuts

A second order cone constraint $\|A_i x + b_i\| \leq c_i^T x + d_i$ is equivalent to

$$x^T \underbrace{(A_i^T A_i - c_i c_i^T)}_{\equiv Q_i} x + 2 \underbrace{(A_i^T b_i - d_i c_i)^T}_{\equiv q_i^T} x + \underbrace{(b_i^T b_i - d_i)}_{\equiv \gamma_i} \leq 0, \quad c_i^T x + d_i \geq 0.$$

If $Q_i \succeq 0$ for all $i = 1, \ldots, N$, then the methods developed in §2 and §3 apply immediately. We are currently investigating two strategies for the case where the matrices Q_i are indefinite:

(a) If Q_i is indefinite then the relaxed set K is still a non-convex quadratically constrained set. Kojima and Kim have recently developed a SOCP-based method for generating good convex approximations for non-convex quadratically constrained sets [7]. Quadratic cuts can be generated from this convex approximation.

(b) By translating and scaling, the feasible region of the mixed integer SOCP can be assumed to be $K^\circ = \{x \mid \|A_i x + b_i\| \leq c_i^T x + d_i, x_k \in \{1, -1\}, k = 1, \ldots, p\}$. Define a relaxation $\tilde{K}$ as follows: $x \in \tilde{K}$ if and only if there exists $Y \succeq xx^T$ such that $Y_{kk} = 1$ for all $k = 1, \ldots, p$, and

$$\begin{aligned}
&\|A_i Y_k + x_k b_i\| \leq c_i^T Y_k + x_k d_i, \quad i = 1, \ldots, N, \quad k = 1, \ldots, p, \\
&\|A_i x + b_i - A_i Y_k - x_k b_i\| \leq c_i^T x + d_i - c_i^T Y_k + x_k d_i, \quad i = 1, \ldots, N, \quad k = 1, \ldots, p,
\end{aligned}$$

where Y_k is the k-th column of Y. Thus, $\tilde{K}$ is described by $2pN$ second-order cone constraints and one semidefinite constraint in $O(n^2)$ variables. Quadratic cuts can be generated by solving the maximum separation problem with respect to the polar set $\tilde{K}^*$. This relaxation is an extension of the relaxation developed by Lovász and Schrijver [9], Balas et al [2], and Sherali and Adams [12].

References

[1] E. Balas. Disjunctive programming. *Annals of Dis. Math.*, 5:3–51, 1979.

[2] E. Balas, S. Ceria, and G. Cornuèjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Math. Prog.*, 58:295–324, 1993.

[3] D. Bienstock. Computational study of a family of mixed integer quadratic programming problems. *Math. Prog.*, 74:121–140, 1996.

[4] M. T. Çezik. *Semidefinite methods for the traveling salesman and other combinatorial problems.* PhD thesis, IEOR Dept., Columbia University, 2000.

[5] G. N. Iyengar and M. T. Çezik. Cutting plane methods for mixed 0-1 semidefinite programs. Submitted to *IPCO VIII*, 2000.

[6] S. Kim and M. Kojima. Second order cone programming relaxations of nonconvex quadratric optimization problems. Technical Report B-363, Dept. of Math. & CS, Tokyo Inst. of Tech., July 2000.

[7] M. Kojima and L. Tunçel. Cones of matrices and successive convex relaxations of nonconvex sets. *SIAM J. Optim.*, 10(3):750–778 (electronic), 2000.

[8] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret. Applications of second-order cone programming. *Linear Algebra Appl.*, 284(1-3):193–228, 1998.

[9] L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. *SIAM Journal of Control and Optimization*, 1(2):166–190, 1991.

[10] Y. Nesterov and A. Nemirovskii. *Interior-point polynomial algorithms in convex programming.* SIAM, Philadelphia, 1993.

[11] E. Phan-huy Hao. Quadratically constrained quadratic programming: some applications and a method for solution. *Z. Oper. Res. Ser. A-B*, 26(3):105–119, 1982.

[12] H. Sherali and W. Adams. A heirarchy of relaxations and convex hull representations for mixed zero-one programming problems. Technical report, Virginia Tech., Blacksburg, VA, 1989.

[13] R. A. Stubbs and S. Mehrotra. A hierarchy of semidefinite relaxations & convex nonlinear valid inequalities for mixed 0-1 convex programming. In *Proc. of INFORMS*, 1997.

[14] R. A. Stubbs and S. Mehrotra. A branch-and-cut method for 0-1 mixed convex programming. *Math. Prog.*, 86:515–532, 1999.

[15] R. A. Stubbs and S. Mehrotra. Generating convex quadratic inequalities for mixed 0-1 programs. Technical report, Dept. of IEMS, Nothwestern Univ., 1999.

[16] L. Vandenberghe, S. Boyd, and S.-P. Wu. Determinant maximization with linear matrix inequality constraints. *SIAM J. on Matrix Anal. and Appl.*, 19:499–533, 1998.