Semidefinite relaxations for the traveling salesman problem

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1 Introduction

The traveling salesman problem (TSP) is motivated by the problem of tour planning for a salesman who wants to visit a collection of cities exactly once and return incurring minimum travel costs. If cities are represented by nodes of a graph and the paths between cities by edges with edge weights representing the travel cost, then every feasible travel plan is a set of edges such that exactly one edge enters and exactly one leaves every node and the graph is connected. Such a collection of edges is called a tour. Therefore, the TSP reduces to the problem of finding the cheapest tour in a weighted graph. Although the problem was introduced in the context of tour planning, the TSP also serves as a model for problems involving sequencing locations, jobs, net-lists, etc. For discussion of the TSP and its applications, see [13].

The TSP is an NP-hard optimization problem; consequently, efficient enumeration is essentially the only solution strategy known yet. NP-hard combinatorial problems are typically solved via linear programming relaxations that are sequentially strengthened by adding cutting planes or by branching on fractional variables. There are several very successful codes that combine this branch-and-cut framework with heuristics and local search to solve the TSP [2, 14].

An alternative approach for solving NP-hard problems is to design approximation algorithms that have polynomial complexity and provide feasible solutions provably close to the optimal. An approximation algorithm is called k-optimal if the the feasible solution it generates is within a factor k of the optimal solution. Currently the best known results for the TSP are a $\frac{3}{2}$-approximation algorithm for instances when the edge weights satisfy the triangular inequality [5, 15], and an $(1+\epsilon)$-approximation algorithm for the Euclidean TSP [3]. It is also known that, unless P=NP, there cannot be a $(1+\epsilon)$ approximation algorithm for general TSP.

Initially approximation algorithms relied on suitably rounding the solutions of linear programming relaxations of the combinatorial problems [17]. Recently there has been a growing interest in semidefinite programs (SDP) as a means for developing relaxations and approximation algorithms for combinatorial optimization problems. Semidefinite relaxations have led to good approximation algorithms for several NP-hard problems such as the max-cut [10], graph coloring [11], vertex cover [12] etc. For a survey see [8, 1].

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Cvetković et al [6] developed the first semidefinite relaxation for the TSP. The key idea was to use algebraic connectivity to impose connectivity. Algebraic connectivity is the condition that the second smallest eigenvalue of the Laplacian matrix is positive [7]. Cvetković et al reformulated this eigenvalue constraint as a semidefinite constraint. It was recently shown that this semidefinite relaxation is weaker than the linear relaxation with all the subtour elimination inequalities [9].

In Section 2, we develop stronger semidefinite relaxations for the TSP by interpreting properties of tours, such as the chromatic number, the number of spanning trees, and translational invariance etc., as constraints on the eigenvalues and determinants of certain graph matrices. These constraints are reformulated as semidefinite constraints to derive the tighter SDP relaxations. We also show that the TSP can be formulated as a 0-1 SDP, i.e. an SDP with binary decision variables.

In Section 3, we extend the well known Chvátal-Gomory cutting plane method for integer linear programming to develop a procedure for generating valid linear inequalities for 0-1 SDPs. Using this procedure and the the associated notion of rank we investigate the relation between linear relaxations and the semidefinite relaxations developed in Section 2. For example, we establish that the subtour elimination inequalities are rank-1 Chvátal-Gomory cuts with respect to the semidefinite constraint imposing algebraic connectivity. This gives an alternative proof for the known result that the semidefinite relaxation developed by Cvetković et al is weaker than the linear relaxation with subtour elimination constraints [9].

We have developed a MATLAB code that incorporates the relaxations proposed in this abstract. The semidefinite programs were solved using SeDuMi [16], a MATLAB based self-dual minimization software. The computational performance of the code on instances from TSPLIB is encouraging. However, it is clear that these relaxations cannot yet compete with the linear-relaxation based methods [2].

The focus of our current research is on the proving bounds on the integrality gaps of the semidefinite relaxations and developing semidefinite programming based approximation algorithms for the TSP. We are also working on extending these relaxations to related problems such as vehicle routing and network design.

2 Spectral Relaxations for the Traveling Salesman Problem

In this abstract $\mathcal{G} = (V, E)$ will denote an undirected graph with n vertices, i.e. $|V| = n$, and edge set $E \subseteq \{(i,j) : 1 \leq i,j \leq n\}$. The matrix $A \in \{0,1\}^{n \times n}$ will denote the symmetric node-node adjacency matrix defined as follows,

$$A(i,j) = \begin{cases} 1 & (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

A graph $T = (V, E_T)$ is called a tour if it is connected and its edge set $E_T = \{(i_k, j_k) : 1 \le k \le n\}$ has the property that for every vertex v there exists a unique k with $i_k = v$ and a unique k' with $j_{k'} = v$. Thus, the corresponding node-node adjacency matrix A_T is feasible for the set of equations,

$$\begin{array}{rcl} A\mathbf{1} & = & 2\mathbf{1}, \\ \mathbf{diag}(A) & = & 0, \\ A_{ij} & \in & \{0,1\}, \quad \forall i,j = 1,\ldots,n. \end{array} \tag{1}$$

Let $C : V \times V \mapsto \mathbf{R}$ be the function that edge (i,j) to the cost C_{ij}. Then, the TSP is equivalent to the following optimization problem.

$$\begin{array}{ll} \text{minimize} & \sum_{i,j=1}^{n} C_{ij} A_{ij}, \\ \text{subject to} & A \text{ is the adjacency matrix of a tour.} \end{array} \tag{2}$$

In the rest of this section we develop efficient relaxations for this optimization problem.

2.1 Semidefinite constraints based on connectivity

The first semidefinite relaxation for the TSP was developed by Cvetković et al [6]. The key idea of this relaxation is to use algebraic connectivity [7] to enforce connectivity. The algebraic connectivity $\alpha(A)$ is the second-smallest eigenvalue of the Laplacian matrix $L(A)$ defined as follows,

$$L(A) = D - A,$$

where D is a diagonal matrix with D_{ii} equal to the degree of the vertex i. Fielder [7] showed that a graph, that is not complete, is connected if and only if $\alpha(A) > 0$.

It is easy to establish that $L(A)$ is positive semidefinite and that the smallest eigenvalue $\lambda_1(L(A)) = 0$. Therefore, $\alpha(A)$ is a concave function of the matrix A [1].

Let C_n be a cycle on n vertices then the eigenvalues of $L(C_n)$ are $1 - \cos(\frac{2\pi k}{n})$ for $1 \le k \le n$. Define h_n to be the algebraic connectivity of C_n, i.e. $h_n = \alpha(C_n) = 1 - \cos\left(\frac{2\pi}{n}\right)$. Since tours are cycles, we have that $\alpha(A_T) \ge h_{n-1}$ is a valid convex constraint.

Using the fact that $\lambda_1(L(A)) = 0$ with $\mathbf{1}$ as the corresponding eigenvector, Cvetkvović et al showed that this convex eigenvalue constraint can be reformulated as a semidefinite constraint. The following lemma is a slightly extended version of the corresponding result in [6].

Lemma 1 ([6]) *Let T be any tour in graph $\mathcal{G} = (V, E)$ with $|V| = n$ and A_T be its adjacency matrix. Then*

$$(2 - h_n)I + \frac{h_n}{n}\mathbf{1}\mathbf{1}' - A_T \succeq 0, \tag{3}$$

where $A \succeq 0$ denotes that A is positive semidefinite. Moreover, if A satisfies (1) and (3) then the second largest eigenvalue $\lambda_2(2I - A) \ge h_n$.

The following corollary immediately follows

Corollary 1 *The TSP is equivalent to the following binary SDP,*

$$\begin{array}{ll} \textit{minimize} & \sum_{i,j=1}^{n} C_{ij}A_{ij}, \\ \textit{subject to} & A\mathbf{1} = 2\mathbf{1}, \quad \mathbf{diag}(A) = \mathbf{0}, \quad A \in \{0,1\}^{n\times n}, \\ & (2-h_n)I + \frac{h_n}{n}\mathbf{1}\mathbf{1}' - A \succeq 0. \end{array} \tag{4}$$

We get the following SDP relaxation by relaxing the binary constraints,

$$\begin{array}{ll} \text{minimize} & \sum_{i,j=1}^{n} C_{ij}A_{ij}, \\ \text{subject to} & A\mathbf{1} = 2\mathbf{1}, \quad \mathbf{diag}(A) = \mathbf{0}, \quad A \in [0,1]^{n\times n}, \\ & (2-h_n)I + \frac{h_n}{n}\mathbf{1}\mathbf{1}' - A \succeq 0. \end{array} \tag{5}$$

Since the Laplacian $L(A) \succeq 0$, and $h_n \to 0$ as $n \to \infty$, it is unlikely that the constraint $\alpha(A) \geq h_n$ would lead to a tight relaxation. In fact, Goemans and Rendl [9] showed that the linear relaxation generated the assignment constraints and the subtour elimination inequalities dominates the semidefinite relaxation (5). Thus, one needs stronger semidefinite constraints for this approach to be successful.

2.2 Semidefinite constraints based on coloring

Any feasible solution A of (5) can be interpreted as the adjacency matrix of a undirected weighted graph on n nodes. In our computational experiments with (5) we found that the optimal solution A^* corresponds to a graph with many extra edges. In other words, the chromatic number of the support graph of A^* is considerably higher than those of tours. In this section, we develop semidefinite constraints that constrain the chromatic number of the feasible graphs.

Recall that Lovász-θ function for the complement $\overline{\mathcal{G}}$ of a graph $\mathcal{G}$ is given by,

$$\begin{array}{rcll} \theta(\overline{\mathcal{G}}) & = & \text{minimize} & \lambda_{\max}(B + \mathbf{1}\mathbf{1}') \\ & & \text{subject to} & B = B', \\ & & & B_{ij} = 0, \qquad \forall (i,j) \in E. \end{array} \tag{6}$$

The Sandwich Theorem due to Lovász states that for any graph $\mathcal{G}$,

$$\omega(\mathcal{G}) \leq \theta(\overline{\mathcal{G}}) \leq \chi(\mathcal{G})$$

where $\omega(\mathcal{G})$ is the size of the largest clique and $\chi(\mathcal{G})$ is the chromatic number of the graph $\mathcal{G}$. The inequalities are tight if the graph $\mathcal{G}$ is perfect. A tour T with even number of vertices is a perfect graph with $\chi(\mathcal{G}) = \omega(\mathcal{G}) = 2$, therefore, $\theta(\overline{T}) \leq 2$, is a valid constraint. We show that this inequality can be reformulated as semidefinite constraint.

To this end, define $\tilde{\theta}(\mathcal{G})$ as follows

$$\begin{array}{rll}\tilde{\theta}(\bar{\mathcal{G}}) = & \text{minimize} & \lambda_{\max}(B + \mathbf{1}\mathbf{1}'), \\ & \text{such that} & B = B', \quad \mathbf{diag}(B) = 0, \\ & & B_{ij} = 0, \qquad\qquad\qquad (i,j) \in E, \\ & & B_{ij} \in \{0, -2\}, \qquad\quad i,j = 1, \ldots, n.\end{array} \tag{7}$$

Since (7) is a restriction of the optimization problem (6), we have that $\theta(\overline{\mathcal{G}}) \leq \tilde{\theta}(\overline{\mathcal{G}})$. It can be easily established that for tours the two functions are equal, i.e. $\theta(\overline{T}) = \tilde{\theta}(\overline{T})$ for any tour T. Thus, $\tilde{\theta}(\overline{T}) \leq 2$, is a valid constraint for all tours on even number of vertices.

Next, we show that this constraint can be formulated as a semidefinite constraint on the adjacency matrix A. Let $0 \leq A \leq \mathbf{1}\mathbf{1}'$ and define $M = \frac{1}{2}(B + 2\mathbf{1}\mathbf{1}')$, then the optimization problem (7) can be rewritten as follows,

$$\begin{array}{rll}\tilde{\theta}(\bar{\mathcal{G}}) = & \text{minimize} & \lambda_{\max}(2M - \mathbf{1}\mathbf{1}'), \\ & \text{such that} & M = M', \quad \mathbf{diag}(M) = \mathbf{1}, \\ & & M \geq A, \\ & & M_{ij} \in \{0, 1\}, \qquad\qquad i,j = 1, \ldots, n.\end{array} \tag{8}$$

Thus, the constraint $\tilde{\theta}(\overline{\mathcal{G}}) \leq 2$ is equivalent to the constraint that there exists M satisfying the constraints in (8) such that $2\mathbf{I} - (2M - \mathbf{1}\mathbf{1}') \succeq 0$. On relaxing the binary constraint on M we get the following semidefinite constraint on the adjacency matrix A.

Lemma 2 *Let A be the adjacency matrix of a tour on even number of vertices. Then there exists M such that*

$$\begin{array}{rcl}2\mathbf{I} + \mathbf{1}\mathbf{1}' - 2M & \succeq & 0, \\ \mathbf{diag}(M) & = & \mathbf{1}, \\ M & \geq & A, \\ M_{ij} & \in & [0,1], \qquad i,j = 1, \ldots, n.\end{array} \tag{9}$$

Thus, we have a new semidefinite constraint for the adjacency matrix A of tours on even number of nodes in terms of auxiliary variable M. This constraint tries to capture the fact that the chromatic number of even tours is 2.

This relaxation for the TSP on even number of vertices can be extended to graphs with an odd number of vertices as follows. Add a dummy vertex 0 and set the costs $C_{0i} = C_{1i}$ and $C_{i0} = C_{1i}$ for all $i = 2, \ldots, n$. Then, adjacency matrices for tours on $V = \{1, \ldots, n\}$ are in one-to-one correspondence with adjacency matrices for tours on $V \cup \{0\}$ such that $A_{01} = A_{10} = 1$. Moreover, this correspondence preserves the cost. Thus, the problem reduces to solving the TSP over even number of nodes.

2.3 Semidefinite constraints from spanning trees

The number of spanning trees rooted at a particular node v in a graph $\mathcal{G}$ is given by $\det(L_v)$ where L_v is the Laplacian matrix L with its row and column corresponding to v is removed. Since there are exactly n spanning trees rooted at every node, it follows that

$$\log\det(2\mathbf{I} - A_v) \leq \log(n), \quad \text{for all } v = 1, \dots, n,$$

where A_v is the adjacency matrix with v-th row and column removed.

The interior-point solvers for semidefinite programs impose the condition $A \succeq 0$ by using the barrier function $\log\det(A)$. Since the spanning tree constraint is of this form, it can be easily incorporated into such interior point solvers. Vandenberghe et al [4] have studied this problem and developed special purpose codes for incorporating such constraints.

2.4 Semidefinite constraints from translational invariance

Let X be the directed adjacency matrix of a tour, i.e.

$$X_{ij} = \begin{cases} 1, & \text{vertex } j \text{ follows } i \text{ in the tour,} \\ 0, & \text{otherwise,} \end{cases}$$

Then, the matrix X^p is also the directed adjacency matrix of a tour for all p such that n and p are mutually prime. Therefore, one can tighten the semidefinite relaxations for tours by introducing constraints for powers of X.

Consider the special case with $p = 2$. The matrix X^2 represents tour for all graphs with odd number of nodes. The undirected adjacency matrix corresponding to X^2 is given by $X^2 + (X')^2$, and therefore the corresponding Laplacian $L(X^2)$ is given by

$$\begin{aligned} L(X^2) &= 2\mathbf{I} - (X^2 + (X')^2), \\ &= XX' + X'X - (X^2 + (X')^2), \\ &= (X - X')(X' - X), \end{aligned}$$

where we use the identity $XX' = \mathbf{I}$.

As before, one would like to constraint that $\alpha(X^2) = \lambda_2(L(X^2)) \geq h_n$, i.e.

$$(X - X')(X' - X) \succeq h_n \mathbf{I} - \frac{h_n}{n}\mathbf{1}\mathbf{1}'$$

Unfortunately this constraint is not convex in X. On the other hand, the constraint

$$\sum_{i=1}^{k} \lambda_{[i]}(L(X^2)) \leq S_k,$$

where $\lambda_{[i]}(L(X^2))$ is the i-th largest eigenvalue of $L(X^2)$ and S_k is the sum of the k largest eigenvalues of cycles is convex in X. It can be rewritten as the following set of constraints,

$$\begin{aligned} kz + \mathbf{Tr}(V) &\leq S_k, \\ zI + V &\succeq (X - X')(X' - X), \\ V \succeq 0. \end{aligned}$$

Using Schur complements this set of constraints can be reformulated as the following semidefinite constraint

$$\begin{array}{rcl} kz + \mathbf{Tr}(V) & \leq & S_k \\ \begin{bmatrix} I & X - X' \\ X' - X & zI + V \end{bmatrix} & \succeq & 0 \\ V & \succeq & 0 \end{array}$$

We are still investigating the contribution of these constraints in reducing the optimality gap.

3 Chvátal-Gomory cutting planes for the TSP

In this section, we extend the Chvátal-Gomory (C-G) procedure developed for integer programs to semidefinite programs with integer variables. We show that the C-G procedure can be used as a tool to investigate the relationship between the semidefinite and polyhedral relaxations of the TSP.

Since the positive semidefinite cone is self-dual, it follows that $A \succeq 0$ if and only if, for all $Z \succeq 0$, $A \bullet Z \geq 0$, where $A \bullet Z = \mathbf{Tr}(AZ)$ is the usual inner product on the space of matrices. Thus, an affine semidefinite constraint,

$$F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0,$$

where $F_0 = F_0', F_i = F_i' \in \mathbf{R}^{m \times m}$, is equivalent to

$$Z \bullet F_0 + \sum_{i=1}^{n} x_i (Z \bullet F_i) \geq 0, \quad \forall Z \succeq 0.$$

In other words, one semidefinite constraint is equivalent to an infinite set of linear inequalities. Consider the following semidefinite program with integer variables,

$$\begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0, \\ & x \in \mathbf{Z}_+^n, \end{array} \tag{10}$$

Using the equivalence above, we develop the following Chvátal-Gomory-type procedure for integer semidefinite programming.

Step 1. For any $Z = Z' \succeq 0$, $Z \bullet F_0 + \sum_{i=1}^n x_i (Z \bullet F_i) \geq 0$ is a valid inequality for (10).

Step 2. Since $x_i \geq 0$, it follows that $F_0 \bullet Z + \sum_{i=1}^n x_i \lceil Z \bullet F_i \rceil \geq 0$.

Step 3. Since $x \in \mathbf{Z}_+^n$, it follows that $\lfloor F_0 \bullet Z \rfloor + \sum_{i=1}^n x_i \lceil Z \bullet F_i \rceil \geq 0$.

The valid inequality

$$\lfloor Z \bullet F_0 \rfloor + \sum_{i=1}^{n} x_i \lceil Z \bullet F_i \rceil \geq 0, \tag{11}$$

is called a Chvátal-Gomory (C-G) cut for the integer semidefinite program (10). As in the case of linear programming, one can combine this cut with other C-G cuts and semidefinite constraint to generate new constraints. One can analogously define the *rank* of the C-G cuts for integer SDPs.

Since the C-G procedure generates valid linear inequalities for integer SDPs – and Corollary 1 establishes that the TSP is a 0-1 SDP – the cutting planes could be used as a tool to characterize the relative strengths of the semidefinite and linear relaxations. As an example, consider the comparison of the sub-tour elimination constraints that impose connectivity and the semidefinite constraint (3) that imposes algebraic connectivity. For a graph with n vertices, the sub-tour elimination constraints are given by

$$\sum_{e \in E(W)} x_e \leq |W| - 1, \quad \forall\, W \subset V, \quad 3 \leq |W| \leq n-1, \tag{12}$$

where $E(W) = \{(i,j) \in E : i, j \in W\}$. We show that all the sub-tour elimination constraints are rank 1 C-G cuts with respect to the semidefinite constraint (3) that imposes connectivity.

Lemma 3 *The sub-tour elimination constraints for the TSP on a graph $\mathcal{G} = (V, E)$ with $|V| = n$ are rank 1 C-G cuts with respect to the semidefinite constraint,*

$$h_n \mathbf{I} - \frac{h_n}{n} \mathbf{1}\mathbf{1}' - A \succeq 0,$$

i.e. for all $W \subset V$ there exists $Z = Z' \succeq 0$ such that the C-G cut of the form (11) generated by Z is precisely $\sum_{e \in E(W)} x_e \leq |W| - 1$.

The result is proved by first showing that for every subtour elimination inequality there exists Z such that the inequality is the corresponding C-G cut and then establishing that the inequality is not a rank 0 C-G cut. This lemma gives an alternate proof for the result that (5) is weaker than the linear relaxation with all the sub-tour elimination inequalities [9].

However, on a positive note, Lemma 3 asserts that the single $(n \times n)$ semidefinite constraint approximates all the exponentially many subtour inequalities in the sense that rank 1 inequalities subsume all these cuts.

Using the semidefinite constraint (9) corresponding to the chromatic number constraint, we prove the following.

Lemma 4 *The 2-matching inequalities for the TSP on a graph $\mathcal{G} = (V, E)$ with $|V| = n$ are rank 1 C-G cuts with respect to the following set of constraints,*

$$\begin{aligned} 2\mathbf{I} + \mathbf{1}\mathbf{1}' - 2M &\succeq 0, \\ \mathbf{diag}(M) &= \mathbf{1}, \\ M &\geq A, \\ M_{ij} &\in [0,1], \qquad i,j = 1,\ldots,n. \end{aligned}$$

We are investigating whether other classes of inequalities, such as the comb inequalities, have efficient representations as semidefinite constraints. We expect that if one is able to represent a class of linear inequalities as rank 1 inequalities of a semidefinite constraint, a relatively small SDP could approximate the effect of an exponential number of inequalities. Such a semidefinite constraint would be particularly interesting if there is no polynomial time separation algorithm for the class of inequalities.

4 Computational results and conclusion

We have conducted some preliminary experiments with the relaxations proposed in this abstract. The relaxations that incorporate the semidefinite constraints corresponding to coloring and connectivity yield an optimality gap of 3% over TSPLIB problems with node size less than 76. This node size restriction is a consequence of memory restrictions in SeDuMi and MATLAB. The run times in these preliminary experiments, although promising, do not compare well with the LP relaxations.

Our current research focus is on characterizing the integrality gap of these relaxations and developing approximation algorithms based on them. We are also working on extending these relaxations to related problems such as vehicle routing. We expect that the insight will lead to semidefinite constraints for other problems involving geometrical constraints.

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