Strong inequalities for capacitated survivable network design problems*

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Abstract

We present several classes of facet-defining inequalities to strengthen polyhedra arising as subsystems of network design problems with survivability constraints. These problems typically involve assigning capacities to a network with multicommodity demands, such that after a vertex- or edge-deletion at least some prescribed fraction of each demand can be routed.

1 Introduction.

Standard network design problems typically involve the installation of "capacities" in the vertices or edges of a graph, in discrete amounts, and at minimum cost, so that multicommodity demands can be routed. A large number of models arise in this manner, resulting in fairly difficult mixed-integer programs. However, cut-and-branch approaches have successfully found near-optimal solutions to realistic problem instances. These algorithms rely on polyhedral inequalities that almost always are "partition" or, especially, "cutset" inequalities (or derivatives thereof) which state that the capacity on the edges (or vertices) of a cut is large enough to handle the demand across the cut. In the simplest case, cutset inequalities take the general form,

$$\sum_{e \in C} x_e \quad \geq \quad \lceil d(C) \rceil \tag{1}$$

where x_e is the capacity of element (vertex or edge) e of the cutset C, and $d(C)$ is the total demand separated by C. Note that this inequality aggregates the multicommodity demands across the cut. See [12]. Also see [4], [6], [7], [13], [14], [17], [16], [21].

The optimal solutions to these problems usually have very sparse support. In recent years, practitioners have focused on adding *survivability* constraints to network design models. These constraints specify that in the event of a vertex or edge deletion (hereafter called a "fault") the remaining capacities should be enough to route at least some prescribed fraction of each demand. (This is in contrast with purely combinatorial connectivity network design problems, see [9], [10], which ask for a graph with desired graph-theoretic connectivity levels between pairs of vertices). Thus, broadly speaking, we have a "no-fault" routing that delivers all the required demands and is used whenever no faults have taken place; this routing is modified in the event of a fault. Several different models arise out of this framework. For example, we may allow

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a rerouting of all commodities to make use of the capacities that survive the fault. Or, we may only allow those commodities that used the deleted vertex or edge to be rerouted. Or, we may "reserve" capacities for each commodity in each vertex (or edge) and specify that these reserved capacities are to be used in the event of a fault. In particular, see [1], [18], [2] for discussions of several alternative models and solution approaches. Also see [3], [15], [20], [11].

It has been observed by various researchers that in the presence of survivability constraints, the inequalities (1) become weak and lead to poor bounds. In fact, survivability constraints will imply a usually stronger right-hand side value than that in (1). In this paper we discuss several polyhedra arising as surrogates when survivability constraints are added to network design problems. In Section 2 we study a basic relaxation that strengthens a single inequality (1), in Section 3 we study a polyhedron arising from more complicated node cuts, in Section 4 we study a mixed-integer model with reserved capacities, and in Section 5 we study a multi-commodity version.

The basic model that we are interested in is abstracted from that arising in network design problems described in [5]. Also see [18]. Here the optimization problem chooses both a no-fault routing and feasible capacities so that the combination is survivable.

In [15] the authors have independently considered a model that arises in a different setting, the line restoration problem. Here a no-fault routing is given (i.e. it is an input) and the problem is to add additional, "spare" capacities, so that in the event of the deletion of an edge $\{u, v\}$ the flows using $\{u, v\}$ can be rerouted between u and v using the spare capacities, and without disturbing the other flows. The primary difference with our model lies in that in [15] the no-fault routing is fixed (in an extreme case, simply changing the no-fault routing could make it survivable without requiring any spare capacities; but there are settings in which the no-fault routing is indeed fixed). Refer to Section 2 for a more detailed description of the mathematical differences between the two models.

2 The simple model.

In this section we consider, for integer $n \geq 2$ and positive integers D, L, the polyhedron $P_n(L, D)$ described by the system

$$\sum_{j \neq i} x_j \geq L, \ \forall i, \ 1 \leq i \leq n \tag{2}$$

$$\sum_{j} x_j \geq D \tag{3}$$

$$x \in Z_+^n \tag{4}$$

In terms of the survivability problem, inequality (3) is a surrogate for a cutset inequality (1), with D equal to the ceiling of the total demand across the cut. Furthermore, inequalities (2) state that in the event of a failure at least L units of capacity are required in the surviving elements of the cut. Thus, this system can be viewed as a strengthening of the cutset inequality.

A more complicated model arises when we do not aggregate the commodities across a cut, and specify separate survivability amounts for each commodity. This leads to the following type of polyhedron. We are given an integer $n \geq 2$, an integer $K > 0$ (the number of commodities), and positive *rational* values D^k, L^k, $k = 1, \cdots, K$. The polyhedron is described by:

$$\sum_{j \neq i} f_j^k \geq L^k, \ \forall i, \ 1 \leq i \leq n \ \forall k, \ 1 \leq k \leq K \tag{5}$$

$$\sum_{j} f_j^k \geq D^k, \ \forall k, \ 1 \leq k \leq K \tag{6}$$

$$\sum_k f_i^k - x_i \quad \leq \quad 0, \;\; \forall i, \; 1 \leq i \leq n \tag{7}$$

$$x \in Z_+^n \qquad f \geq 0 \tag{8}$$

Here x_i is the capacity installed in element i of the cut, and f_i^k is the amount of capacity on i "reserved" for commodity k. Inequality (5) states that this capacity allocation is survivable, (6) is a demand inequality per commodity, and inequality (7) states that the allocation is feasible.

Having separate demand inequalities (6) per commodity makes this polyhedron more complex. It is true only in special cases that its projection onto the space of the x variables is $P_n(\lceil \sum_k L^k \rceil, \lceil \sum_k D^k \rceil)$. For example, this holds if there is a multiplier λ such that $L^k = \lambda D^k$ for all k (a reasonable assumption). It also holds under a number of more technical assumptions concerning the relationship between the D^k and L^k, e.g. if $D^k \leq \frac{n-1}{n-2} L^k$ for all k. Also, there is a polynomial-time algorithm for determining a feasible (x, f) with minimum $\sum_i x_i$ (see [8]). In Sections 4 and 5 we examine some relaxations of this polyhedron.

In the rest of this section we first describe the extreme points of $P_n(L, D)$ (there are at most $2n - 3$ up to permutation) and then characterize the facets of $P_n(L, D)$. We will show that any facet is of the form (2), or a nonnegativity condition, or has the form

$$\sum_{j \in I} x_j + \alpha \sum_{j \notin I} x_j \quad \geq \quad R \tag{9}$$

where $I \subseteq \{1, \cdots, n\}$ and $\alpha \geq 1$ (possibly $|I| = n$) and can be obtained by sequential lifting.

Brightwell and Shepherd [20] show how to solve a linear program over $P_n(L, D)$ in polynomial time.

Magnanti and Wang [15] have independently studied a polyhedron similar to $P_n(L, D)$: they omit inequality (3) but allow different right-hand sides for each inequality (2); and present a system of inequalities similar to ours that describes their polyhedron (and thus, ours). Our techniques for characterizing the extreme points of $P_n(L, D)$, and its facets as sequentially lifted inequalities (in particular, Lemma 2.2, Lemma 2.8 and Proposition 2.11) are very different from those used in [15] and are critical for the other models that we study. It appears that these techniques can be adapted to study the polyhedron in [15].

2.1 The inequalities

Our first result considers the minimum capacity allocation.

Theorem 2.1 *For any $m \geq 2$ the inequality*

$$\sum_j x_j \quad \geq \quad \max\{\lceil \frac{m}{m-1} L \rceil, D\} \tag{10}$$

is facet-defining for $P_m(L, D)$ if and only if either $D > \lceil \frac{m}{m-1} L \rceil$ or if $\frac{L}{m-1} \notin Z$.

Proof. This inequality is obtained by adding up all inequalities (2) and observing (3). It is straightforward that the conditions are necessary (or else (10) is implied by all the inequalities $\sum_{j \neq i} x_j \geq L$).

Let F denote the set of points satisfying (10) with equality. Suppose first that $\lceil \frac{m}{m-1} L \rceil \geq D$. Write $L = p(m-1) + r$, where $0 < r < m - 1$. Consider a point x^* in Z_+^m with $m - 1 - r$ entries of value p and $r + 1$ entries of value $p + 1$. The sum of the $m - 1$ smallest entries of x^* is $(m - 1 - r)p + r(p + 1) = L$, and the sum of all entries equals $L + p + 1 = \lceil \frac{m}{m-1} L \rceil$; consequently, $x^* \in P_m(L, D) \cap F$.

We now show that using permutations of this vector one obtains a family of m affinely independent points in $P_m(L, D) \cap F$. For $1 \leq i \leq r + 2$, let v^i be the vector such that $v_j^i = p + 1$, for every $j \leq r + 2$ with $i \neq j$, and $v^i_j = p$ otherwise. For $r+2 \leq i \leq m$ let v^i be the vector such that $v^i_j = p+1$, if $j \leq r$ or if $j = i$, and $v^i_j = p$ otherwise. Let V be the matrix whose rows are the v^i. Then it can be seen that the $(r+2)-$ principal minor of V is invertible, and that the submatrix consisting of the intersection of the last $m-r-2$ rows and the last $m-r-2$ columns is also invertible. Hence V is invertible, as desired.

The case when $D > \lceil \frac{m}{m-1} L \rceil$ is similar and will be omitted.

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Consider the related polyhedra $Q_n(L,K)$ given by

$$\sum_{j \neq i} x_j \geq L, \ \forall i, \ 1 \leq i \leq n \tag{11}$$

$$\sum_j x_j = K \tag{12}$$

where K, L are positive integers. Note that we do not insist on integrality of x.

Suppose x^* is an extreme point of $Q_n(L,K)$. If x^* has $m < n$ positive components, then it is clear that the restriction of x^* to its positive components is (up to renaming) an extreme point of $Q_m(L,K)$ ($n-m$ of the inequalities (11) are redundant).

Furthermore, if $x^*_j > 0 \ \forall j$, then $n-1$ of the constraints (11) are tight. Consequently, in this case $n-1$ of the components of x^* take the value $K-L$ and the remaining component takes the value $(n-1)L - (n-2)K$ (and also $(n-1)L - (n-2)K \leq K-L$) Thus $Q_n(L,K)$ is an integral polyhedron. We note that when $n=2$ the above implies that $K \geq 2L$, and the extreme points of $Q_2(L,K)$ are $(K-L, L)$ and $(L, K-L)$.

Now any extreme point of $P_n(L,D)$ must be an extreme point of $Q_n(L,K)$ for appropriately chosen K (namely, the sum of variables). Putting all of this together, we have

Lemma 2.2 *$P_n(L,D)$ has an extreme point x^* with $2 < m \leq n$ positive components precisely when*

$$\max\{\lceil \frac{m}{m-1} L \rceil, D\} < \frac{m-1}{m-2} L \tag{13}$$

Further, denoting $K^* = \sum_j x^*_j$, we have either $K^* = \max\{\lceil \frac{m}{m-1} L \rceil, D\}$ or $K^* = \lfloor \frac{m-1}{m-2} L \rfloor$, $m-1$ of the components of x^* are equal to $K^* - L$, one component (the smallest positive) is equal to $(m-1)L - (m-2)K^*$, and the remaining are zero. When K^* takes the higher limiting value, L is not a multiple of $m-2$.

Proof Sketch. If $m > 2$ and K^* takes a value strictly between the two limiting values in (13) then (using the fact that $m-1$ components take value $K^* - L$ and one takes value $(m-1)L - (m-2)K^*$) it can be shown that x^* is the half-sum of an extreme point of $Q_m(L, K^* - 1)$ and an extreme point of $Q_m(L, K^* + 1)$. Note that when L is a multiple of $m-2$, the limiting case $K^* = \lfloor \frac{m-1}{m-2} L \rfloor$ only yields a point with $m-1$ positive components (since $(m-1)L - (m-2)K^* = 0$). ■

Remark 2.3 *For a complete proof of the above result, see [8]. In what follows we will only need the fact that the points are integral and in $P_m(L,D)$.*

Remark 2.4 *Suppose condition (13) is violated because $D \geq \frac{m-1}{m-2} L = L + \frac{L}{m-2}$. Then every extreme point has strictly fewer than m positive components. The other case when (13) is violated occurs when $\lceil \frac{L}{m-1} \rceil \geq \frac{L}{m-2}$. If we write $L = q * (m-1) + r$, where $q \geq 0$ and $0 \leq r < m-1$, this case occurs when $r > 0$ and $r + q \leq m-2$. Note that in this case*

$$\lceil \frac{m}{m-1} L \rceil = L + q + 1 = \lceil \frac{m-1}{m-2} L \rceil.$$

Moreover, if there is no extreme point with m positive components, but there is one with $m+1$, a similar analysis shows that $\lceil \frac{L}{m} \rceil = \lfloor \frac{L}{m-1} \rfloor$ and hence (up to permutation) there is a unique extreme point with $m+1$ positive components.

Further, note that if the conditions of Theorem (2.1) apply, an integral point x with m positive coordinates and sum of coordinates K satisfies $K > \frac{m}{m-1}L$, and thus $K - L > (m-1)L - (m-2)K$, so the smallest positive coordinate is strictly smaller than the others.

Finally, note that extreme points with two positive components always exist: one component of value L, and the other of value $\max\{L, D-L\}$.

Example 2.5 *Suppose $L = 57$, $D = 60$, $n = 30$. We have $L + \lfloor \frac{L}{20} \rfloor < D$. Hence there is no extreme point with 22 or more positive components. Moreover, $\frac{L}{11} = 5.18$, $\frac{L}{12} = 4.75$, $\frac{L}{13} = 4.38$, $\frac{L}{14} = 4.07$, $\frac{L}{15} = 3.8$ and consequently there are extreme points with 13 and 16 positive components, none with 14 or 15. Similarly, it may be checked that there are is an extreme point with 11 positive components, but none with 12.*

Next we turn our attention to lifting inequalities of type (10), i.e. lifting an inequality $\sum_{j \le m} x_j \ge R$ (with R as in Theorem 2.1) which is facet-defining for $P_m(L,D)$, to an inequality

$$\sum_{j \le m} x_j + \alpha x_{m+1} \quad \ge \quad R, \tag{14}$$

facet-defining for $P_{m+1}(L,D)$. Clearly we must have $\alpha \ge 1$ for this inequality to be valid. Any extreme point x^* of $P_{m+1}(L,D)$ which defines α (i.e. it is tight for (14)) must therefore have $x^*_{m+1} = \min_i\{x^*_i\}$ since permuting components yields another point in $P_{m+1}(L,D)$. Thus, if $P_{m+1}(L,D)$ has an extreme point with $m+1$ positive components, then the computation of α reduces to checking the two different extreme points of $P_{m+1}(L,D)$ where x^*_{m+1} attains the minimum value; this value as described in Lemma 2.2 being of the form $mL - (m-1)K$, where $K = \sum_{j \le m+1} x^*_j$.

On the other hand, if $P_{m+1}(L,D)$ has no extreme point with $m+1$ positive components, then $\alpha = 1$. Further, we may also be able to lift (14) to a facet of $P_{m+p}(L,D)$ (for a certain $p > 0$) by simultaneously lifting $x_{m+1}, \cdots, x_{m+p}$.

Example 2.6 *Suppose $L = 57$, $D = 60$. Then the inequality*

$$\sum_{j \le 11} x_j \quad \ge \quad 63 \tag{15}$$

is facet-defining for $P_{11}(L,D)$. However, as described in Example 2.5 there is no extreme point of $P_{12}(L,D)$ with 12 positive components; hence

$$\sum_{j \le 11} x_j + x_{12} \quad \ge \quad 63$$

is facet-defining for $P_{12}(L,D)$. This inequality can be lifted, yielding the facet

$$\sum_{j \le 11} x_j + x_{12} + \frac{3}{2} x_{13} \quad \ge 63 \tag{16}$$

of $P_{13}(L,D)$, tight for the point $(5,5,\cdots,5,2)$. On the other hand, we can lift (15) to

$$\sum_{j \le 11} x_j + \frac{8}{7} x_{12} + \frac{8}{7} x_{13} \quad \ge 63 \tag{17}$$

which is also facet-defining for $P_{13}(L,D)$, and tight for $(5,5,\cdots,5,2)$ and $(5,5,\cdots,2,5)$.

In order to prove our lifting result, we need the following.

Proposition 2.7 *Let $a > 0$, $b > 0$ and $c > 0$ and $d > 0$ be such that $\frac{a}{b} \geq \frac{c}{d}$. Then for $b > dz$ the function*

$$f(z) \quad = \quad \frac{a - cz}{b - dz}$$

is nondecreasing in z, and is strictly increasing if $\frac{a}{b} > \frac{c}{d}$.

Now we have:

Lemma 2.8 *Given $m \geq 2$, let $p > 0$ be smallest such that $P_{m+p}(L, D)$ has an extreme point with $m + p$ positive components. If we start with the facet*

$$\sum_{j \leq m} x_j \quad \geq \quad R \tag{18}$$

of $P_m(L, D)$, and we simultaneously lift $x_{m+1}, \cdots, x_{m+p}$ we obtain the face

$$\sum_{j \leq m} x_j + \alpha \sum_{m+1}^{m+p} x_j \quad \geq \quad R \tag{19}$$

of $P_{m+p}(L, D)$, where

$$\alpha \quad = \quad \frac{R - m\lfloor \frac{L}{m+p-2} \rfloor}{L - (m-1)\lfloor \frac{L}{m+p-2} \rfloor} \quad = \quad \frac{r+1}{r}. \tag{20}$$

where $r = L - (m-1)\lfloor \frac{L}{m-1} \rfloor$. Inequality (19) defines a facet of $P_{m+p}(L, D)$ if and only if either $p = 1$ or $L < (m+p-1)\lfloor \frac{L}{m+p-2} \rfloor$. Further, if $p \geq 2$,

$$\alpha \quad < \quad m\frac{R - L}{L}. \tag{21}$$

Proof. First note that L cannot be an integer multiple of $m + p - 2$. For otherwise, if, say $L = q(m+p-2)$, then applying Lemma 2.2 to points with $m + p$ positive components yields $D < L + q$. Consequently $\max\{\lceil \frac{m+p-1}{m+p-2} L \rceil, D\} = L + q < L + q + \frac{q}{m+p-3} = \frac{m+p-2}{m+p-3} L$ and there is an extreme point with $m + p - 1$ positive components. By definition, we must thus have $p = 1$. But in that case, (18) is not facet-defining because $\frac{L}{m-1} = q \in Z$ and $D < L + q$. So $\frac{L}{m+p-2} \notin Z$.

To obtain the expression (20) for α we use Proposition 2.7. Where H is either $\lfloor \frac{L}{m+p-2} \rfloor$ or $\lceil \frac{L}{m+p-1} \rceil$, the lifting coefficient satisfies an equation of the form,

$$mH + \alpha((p-1)H \; + \; L \; - (m+p-2)H) \quad = \quad R, \tag{22}$$

that is,

$$\alpha \quad = \quad \frac{R - mH}{L - (m-1)H}, \tag{23}$$

The first choice for H is the larger, by our assumption on p, and since $\frac{R}{L} > \frac{m}{m-1}$ and $L > (m-1)\lfloor \frac{L}{m+p-2} \rfloor$, Proposition 2.7 yields the stated value for α. Notice that we must have $D < L + \lfloor \frac{L}{m-1} \rfloor$, or else there are no extreme points with more than m positive components. So $R = L + \lceil \frac{L}{m-1} \rceil = L + \lceil \frac{L}{m+p-2} \rceil$, and (as in Remark 2.4) we have that $\lfloor \frac{L}{m+p-2} \rfloor = \lfloor \frac{L}{m-1} \rfloor$, yielding $\alpha = \frac{r+1}{r}$. Regarding when (19) is facet-defining, the case $p = 1$ is clear. When $p > 1$, the point in $P_{m+p}(L, D)$ that defines α has $m + p - 1$ components of value $\lfloor \frac{L}{m+p-2} \rfloor$ and one component of value $L - (m + p - 2)\lfloor \frac{L}{m+p-2} \rfloor$; moreover this last value is strictly smaller than $\lfloor \frac{L}{m+p-2} \rfloor$ when the last condition in the hypothesis holds. Using this fact, it is possible to construct p new affinely independent points on the face (19), all of which have $m + p$ positive components. If on the other hand the condition does not hold the face is not facet-defining in $P_{m+p}(L, D)$ as all extreme points on it satisfy $x_{m+1} = \cdots = x_{m+p}$. The upper bound on α in the case $p \geq 2$ follows from another application of Proposition 2.7. ∎

Example 2.9 *Suppose $m = 7$ and $L = 16 = D$. Then $\sum_{j \leq 7} x_j \geq 19$ is facet-defining for $P_7(L, D)$. There is no extreme point of $P_9(L, D)$ with eight positive components, but there is one with nine (so $p = 2$) and we obtain the face $\sum_{j \leq 7} x_j + \frac{5}{4}(x_8 + x_9) \geq 19$. However, the only extreme points that are tight for this inequality are $(3, 3, 3, 3, 3, 3, 1, 0, 0)$ (and all permutations of the seven first entries) and $(2, 2, 2, 2, 2, 2, 2, 2, 2)$ and thus we do not have a facet.*

The above Lemma considers the first lifting coefficient. As we continue lifting, subsequent coefficients will be at least as large as the first. However, we will show below that the two types of lifting as shown in Example 2.6 yield all the facets with all coefficients positive.

Example 2.10 *Suppose as in Example 2.6 that $L = 57$, $D = 60$. If we try to lift (17) to $P_{16}(L, D)$, we know that $\frac{8}{7}$ is a lower bound on the lifting coefficient. However there is only one extreme point with 16 positive components, all of them equal to 4. Using $\frac{8}{7}$ as a guess for the lifting coefficient, this extreme point will achieve a left-hand side value of $(11 + 5 \times \frac{8}{7}) \times 4$ which is greater than 63. Thus, we conclude that the lifting coefficient is again $\frac{8}{7}$.*

Similarly, if we lift (16) to $P_{16}(L, D)$, since $(12 + 4 \times \frac{3}{2}) \times 4 > 63$ again the lifting coefficient repeats.

Using the above results, we show now as a corollary that once we have performed one lifting, any subsequent liftings will yield the same coefficient:

Proposition 2.11 *Let $m \geq 2$, $p > 0$, $\gamma \geq 1$ and suppose the inequality*

$$\sum_{j \leq m} x_j + \gamma \sum_{m+1}^{m+p} x_j \quad \geq \quad R \tag{24}$$

is valid for $P_{m+p}(L, D)$ and there are extreme points of $P_{m+p}(L, D)$ with $m + p$ positive components. Then the inequality

$$\sum_{j \leq m} x_j + \gamma \sum_{m+1}^{n} x_j \quad \geq \quad R \tag{25}$$

is valid for $P_n(L, D)$ for any $n > m + p$. If $\gamma > 1$ it is strict for points with more than $m + p$ positive components.

Proof. Since $\gamma \geq 1$, the symmetry of $P_n(L, D)$ guarantees that the validity of (25) only needs to be checked for points x^* with $x_i^* > 0$, $1 \leq i \leq m + p$, and at least one more positive component. If, say, there are $t \geq 1$ additional positive components, the check amounts to verifying that for appropriate K (as before, $= \sum_j x_j^*$)

$$m(K - L) + \gamma(p + t - 1)(K - L) + \gamma((m + p + t - 1)L - (m + p + t - 2)K) \quad \geq \quad R \tag{26}$$

This is the same as checking if

$$m(K - L) + \gamma(mL - (m - 1)K) \quad \geq \quad R, \tag{27}$$

or in other words

$$\gamma \quad \geq \quad \frac{R + mL - mK}{mL - (m-1)K}. \tag{28}$$

To check this, notice that by assumption (24) is valid for $P_{m+p}(L, D)$ and there is an extreme point y of this polyhedron with $m + p$ positive components. If $\sum_j y_j = \bar{K}$, then

$$R \quad \leq \quad m(\bar{K} - L) + \gamma(p-1)(\bar{K} - L) + \gamma((m+p-1)L - (m+p-2)\bar{K}), \tag{29}$$

in other words,

$$\gamma \quad \geq \quad \frac{R + mL - m\bar{K}}{mL - (m-1)\bar{K}}. \tag{30}$$

Since $\bar{K} \geq K$ (because $K \leq L + \lfloor \frac{L}{m+p+t-2} \rfloor \leq L + \lceil \frac{L}{m+p-1} \rceil \leq \bar{K}$) we will have that (30) implies (28) if the function

$$f(z) \quad = \quad \frac{R + mL - mz}{mL - (m-1)z}, \tag{31}$$

satisfies the condition of Proposition 2.7 with $a = R + mL$, $b = mL$ and $c = m$ and $d = m - 1$. This follows from $R \geq L + \frac{L}{m-1}$.

Finally, if $\gamma > 1$ it is clear that we must have strict validity of (25) for points with more than $m + p$ positive components: either $\bar{K} > K$ and we use $f'(z) > 0$, or $\bar{K} = K$ and the first $m + p$ terms of the left-hand side of (26) dominate the terms of the right-hand side of (29). ∎

Corollary 2.12 *Let $I \subseteq \{1, \cdots, n\}$ with $|I| = m \geq 2$, and $p = p(m) > 0$ be smallest such that $P_{m+p}(L, D)$ has an extreme point with $m + p$ positive components, and let $\alpha = \alpha(m, p)$ denote the coefficient obtained when lifting*

$$\sum_{j \in I} x_j \quad \geq \quad R$$

to

$$\sum_{j \in I} x_j + \alpha \sum_{j \notin I} x_j \quad \geq \quad R. \tag{32}$$

Then (32) defines a facet of $P_n(L, D)$ if $n \geq m + p$.

Proof Sketch. The inequality is valid by Proposition 2.11 and through a technique similar to that used in the proof of Theorem 2.1 it is shown to be facet-defining.∎

The following analogue of Proposition 2.11 will be useful later (we omit its proof which is similar):

Proposition 2.13 *Let $m \geq 2$, $1 \leq \delta < \gamma$ and suppose that the inequality*

$$\sum_{j \leq m} x_j + \delta x_{m+1} + \gamma \sum_{m+2}^{m+p} x_j \quad \geq \quad R \tag{33}$$

($p \geq 2$) is valid for $P_{m+p}(L, D)$, that there is an extreme point of $P_{m+p}(L, D)$ with $m+p$ positive components and that $\delta \leq m(R - L)/L$. Then the inequality

$$\sum_{j \leq m} x_j + \delta x_{m+1} + \gamma \sum_{m+2}^{n} x_j \quad \geq \quad R \tag{34}$$

is valid for $P_n(L, D)$ for any $n \geq m + p$, and is strictly valid for points with more than $m + p$ positive components if $\delta < m(R - L)/L$

■

We have now shown how to construct facets of the form (32). In what follows, given $I \subseteq \{1, \cdots, n\}$ with $|I| = m \geq 2$, and $p = p(m) > 0$ smallest such that $P_{m+p}(L, D)$ has an extreme point with $m + p$ positive components, the inequality (9) (with $\alpha = \alpha(m, p)$ the lifting coefficient as defined in Lemma 2.8) will be denoted by $F(I, p, n)$. We note that for $p > 1$ and $n > m+p$, $F(I, p, n)$ is facet-defining even if $F(I, p, m+p)$ is not. Also we note that through a standard argument inequalities of the form (2) or $x_j \geq 0$ define facets.

In the rest of this section we show that we have all the facets. We will assume that we have a facet

$$\sum_j c_j x_j \quad \geq \quad d \tag{35}$$

of $P_n(L, D)$. Clearly $c_j \geq 0$ for all j and therefore $d \geq 0$. A *tight* point will be an extreme point of $P_n(L, D)$ that satisfies (35) with equality.

Proposition 2.14 *Suppose the inequality $\sum_j c_j x_j \geq d$ defines a facet, and $c_i = 0$ for some i. If $d = 0$ then the inequality is some inequality $x_j \geq 0$, and if $d > 0$ then the inequality is $\sum_{j \neq i} x_j \geq L$.*

Proof. The case $d = 0$ is easy.

Suppose $d > 0$. If y is a tight point, then we must have $\sum_{j \neq i} y_j = L$. Otherwise, by slightly decreasing any positive y_j with $j \neq i$ and $c_j > 0$ (which must exist since $d > 0$) and increasing y_i by the same amount, we obtain a new feasible point that violates (35). Thus (35) is one of the inequalities (2). ■

Theorem 2.15 *Suppose the inequality $\sum_j c_j x_j \geq d$ defines a facet, and $c_j > 0$ for all j. Then the inequality is one of the inequalities $F(I, p, n)$ or the inequality (10).*

Proof. Without loss of generality assume $c_1 \leq c_2 \cdots \leq c_n$. In the proof we will make use of the following observation: if $c_i < c_j$, then for any tight y we must have $y_i \geq y_j$ (else exchanging them yields a point that violates (35)).

If $c_1 = c_n$ then clearly (35) is a multiple of (10). Thus, let $m \geq 1$ be smallest such that $c_1 < c_{m+1}$. If $m = 1$ then as just noted any tight y satisfies $y_1 \geq y_j$ for all j, and (through our characterization of extreme points) we have $\sum_{j \geq 2} y_j = L$. Consequently (35) is not a facet.

Hence $m \geq 2$. Through a similar argument, if every tight y satisfies $\sum_{j \geq m+1} y_j > 0$, then the above observation and the structure of the extreme points (i.e., all positive values equal except possibly one, the minimum) imply that, again, $y_1 \geq y_j$ for all j, and we have the same contradiction as in the preceding paragraph.

Consequently, there is a tight y with $y_{m+1} = y_{m+2} = \cdots = y_n = 0$. Rescaling the coefficients c_j and d so that $c_1 = 1$ (but using the same names) we have that

$$\sum_{j \leq m} y_j \quad = \quad d,$$

and consequently $d = \max\{\lceil \frac{m}{m-1} L \rceil, D\}$ (d cannot be larger than this number or (35) would not be valid). In the rest of the proof we show that $c_{m+1} = \cdots = c_n$ and this common value equals a lifting coefficient, so that (35) is one of the inequalities $F(I, p, n)$.

Let $k > 0$ be smallest such that there is a tight point with $m + k$ positive components. Let $p = p(m)$ be smallest such that there is an extreme point x^* with $m + p$ positive components, and let α be the coefficient obtained by simultaneously lifting $\sum_{j \leq m} x_j \geq d$ to $x_{m+1}, \cdots, x_{m+p}$. If $k = 1$, then $p = 1$ and by Proposition

2.11 (35) and $F(I,1,n)$ are identical. If $c_{m+1} \geq \alpha$ then $F(I,p,n)$ dominates (35). So $k > 1$ and $c_{m+1} < \alpha$. Consequently, $p \geq 2$.

Write $\beta = c_{m+2}$. Since $c_{m+1} < \alpha$ we have $\alpha < \beta$. Further, notice that if $k > 2$ we must have $c_{m+2} = \cdots = c_{m+k} = \beta$, else by definition of k every tight point y satisfies $y_{m+1} = y_{m+2}$.

Next, if $p < k$, then by definition of k, x^* is not tight. Hence reducing coefficients $c_{m+2}, \cdots c_{m+p}$ from β to a slightly smaller value $\gamma > \alpha$, we obtain a new inequality

$$\sum_{j \leq m} x_j + c_{m+1} x_{m+1} + \gamma \sum_{m+2}^{m+p} x_j \quad \geq \quad d \tag{36}$$

valid for $P_{m+p}(L,D)$. Since $p \geq 2$, and by Lemma 2.8 $\alpha < m\frac{R-L}{L}$; an application of Proposition 2.13 now shows that the inequality

$$\sum_{j \leq m} x_j + c_{m+1} x_{m+1} + \gamma \sum_{m+2}^{n} x_j \quad \geq \quad d \tag{37}$$

is valid for $P_n(L,D)$ and strictly dominates (35), a contradiction.

Hence $p = k$ and another application of Proposition 2.13 yields that (a) $c_{m+p+1} = \cdots = c_n = \beta$, and (b) inequality (35) has no tight points with more than $m+p$ positive components.

As detailed in Remark 2.4 there is a unique extreme point of the form $z = (z_1, z_2, \cdots, z_{m+p-1}, z_{m+p}, 0, \cdots, 0)$ with all $z_i > 0$, which consequently is on both faces (35) and $F(I,p,n)$. If $n = m+p$ this proves that the two facets are the same (they contain the same extreme points). If $n > m+p$ then all points of the form $(z_1, z_2, \cdots, z_{m+p-1}, 0, \cdots, z_{m+p}, 0, \cdots, 0)$, i.e, obtained by exchanging z_{m+p} with one of the 0 entries, are tight, and thus are the only tight points with more than m positive entries. Since these points are also on the face defined by $F(I,p,n)$, we conclude that $F(I,p,n)$ and (35) define the same facet ■

To conclude this section, we mention that if the definition of $P_n(L,D)$ is changed so that *one* of the inequalities (2) is omitted, i.e., if the underlying model is changed so that one of the elements of the cutset "does not fail" then one obtains results that are very similar to those presented above. For details see [8].

3 An extension of the basic model to node deletions

The basic polyhedron studied above was motivated by the need to strengthen a "cutset" inequality $\sum_{e \in C} x_e \geq \lceil d(C) \rceil$ when the elements of the cutset C may be individually deleted. The results in the previous section apply when C is either an edge-cutset, or a vertex-cutset whose elements do not generate demand.

As a generalization of the simple model, consider the situation that arises when we have an edge-cutset C, we install capacities in the edges of C, and we want these capacities to handle deletions of single vertices incident with the edges of C, assuming that these vertices do not generate demand. This model can be described as follows. We are given a bipartite graph G with edge-set $E(G)$, and positive integers L and D. For a vertex u let $\delta(u)$ denote the set of edges incident with u. We are interested in the polyhedron $B(G,L,D)$ defined by

$$\sum_{e \notin \delta(u)} x_e \quad \geq \quad L, \ \forall \text{ vertices } u, \tag{38}$$

$$\sum_{e \in E(G)} x_e \quad \geq \quad D \tag{39}$$

$$x \in Z_+^n \tag{40}$$

i.e. the sum of capacities of all edges not incident with any particular vertex is at least L. Note that when $E(G)$ is a matching this polyhedron is one of the polyhedra $P_n(L,D)$.

Given such a model, we can obtain a tight lower bound on the sum of all capacities. Here and below, ν is the cardinality of a maximum matching in G. We assume $\nu \geq 2$ or else the polyhedron is empty.

Theorem 3.1 *The inequality*

$$\sum_{e \in E(G)} x_e \quad \geq \quad \max\{\lceil \frac{\nu}{\nu-1} L \rceil, D\} \tag{41}$$

is facet-defining for $B(G,L,D)$ if and only if either $D > \lceil \frac{\nu}{\nu-1} L \rceil$ or if $\frac{L}{\nu-1} \notin Z$. ∎

To prove this and other facts, we will need the following. Let H denote a minimum cardinality vertex-cover of G, i.e H is a set of vertices that intersects every edge, of minimum cardinality. As is well-known, $|H| = \nu$. (Thus, we can obtain (41) by adding inequalities (38) over the elements of H.) An $edge-assignment$ will be a function that maps each edge of G to any of its endpoints that is in H.

Proposition 3.2 *Let H be a minimum-cardinality vertex-cover of G, say $H = \{u_1, \cdots, u_\nu\}$. Consider any valid inequality $\sum_{i \leq \nu} a_i x_i \geq R$ for $P_\nu(L,D)$, and for $1 \leq i \leq \nu$ denote $\bar{a}_{u_i} = a_i$. Then for any edge-assignment λ the inequality*

$$\sum_{e \in E(G)} \bar{a}_{\lambda(e)} x_e \quad \geq \quad R \tag{42}$$

is valid for $B(G,L,D)$.

Proof. Note that for $u \in H$ the set of edges e with $u = \lambda(e)$ is non-empty: any maximum matching will have one edge with that property. For $1 \leq i \leq \nu$ let w_i be the sum of x_e taken over those e with $\lambda(e) = u_i$. Then $x \in B(G,L,D)$ implies $w \in P_\nu(L,D)$. This completes the proof, because the left-hand side of (42) is $\sum a_i w_i$. ∎

Proof of Theorem 3.1. Let M be a maximum matching of G. Then clearly the inequality $\sum_{e \in M} x_e \geq \max\{\lceil \frac{\nu}{\nu-1} L \rceil, D\}$ is facet-defining for $B(M,L,D)$ (by Theorem 2.1). To prove that (41) defines a facet of $B(G,L,D)$ it suffices to provide, for each edge $f \in E(G) - M$, an integral point x in the face (41) with $x_f > 0$ and $x_g = 0$ for any $g \in E(G) - M - f$.

Let $m \leq \nu$ be largest such that there is an extreme point y of $P_\nu(L,D)$ with m positive components. As discussed before $\lceil \frac{L}{m-1} \rceil = \lceil \frac{L}{\nu-1} \rceil$. Further, we may choose y so that it satisfies $\sum_j y_j = \max\{\lceil \frac{m}{m-1} L \rceil, D\} = R$, and by Lemma 2.2 (also see Remark 2.4) $R > \frac{m}{m-1} L$. Thus the smallest positive component of y, without loss of generality y_m, is strictly smaller than the common value of the other positive components.

Number the elements of M so that the first m are $e(1), \cdots, e(m)$. Suppose $f \notin M$. If f has a common vertex with $e_k \in M$ but no other common vertices with elements of M, then $M - e_k + f$ is a matching, and the point x defined by $x_{e(i)} = y_i$ for $i \neq k$, and $x_f = y_k$ is on the face (41).

Suppose now that f intersects *two* of the edges of M, without loss of generality $e(1)$ and $e(2)$. Now if $m = 2$, then y is of the form $(D-L, L)$ (with $D > 2L$). Then setting $x_{e(1)} = D-L-1$, $x_{e(2)} = L$, and $x_f = 1$ again yields the desired x. For $m > 2$, we set $x_{e(1)} = y_1 - 1$, $x_{e(2)} = y_2 - 1$, $x_{e(i)} = y_i$ for $3 \leq i \leq m-1$, $x_{e(m)} = y_m + 1$, and $x(f) = 1$. It is clear that

$$\sum_{e \in M+f} x_e \quad = \quad \sum_j y_j \tag{43}$$

and further, for any vertex u which is not an endpoint of $e(m)$ inequality (38) is satisfied. If on the other hand u is an endpoint of $e(m)$, then the left-hand side of (38) is exactly $\sum_{j<m} y_j - 1 \geq L$, since we had that y_m was strictly smaller than the other components of y. Thus $x \in B(G, L, D)$, and is on the face defined by (41), as desired. ■

Theorem 3.1 is a special case of the following, proved in [8] using results from the previous section:

Theorem 3.3 *Let H be a minimum-cardinality vertex-cover of G, say $H = \{u_1, \cdots, u_\nu\}$. Consider any facet-defining inequality $\sum_{i \leq \nu} a_i x_i \geq R$ for $P_\nu(L, D)$, and for $1 \leq i \leq \nu$ denote $\bar{a}_{u_i} = a_i$. Consider an assignment λ where for an edge $e = \{u, v\}$, $\lambda(e) = u$ if $u \in H$ and either $v \notin H$ or $\bar{a}_u \leq \bar{a}_v$. Then*

$$\sum_{e \in E(G)} \bar{a}_{\lambda(e)} x_e \geq R \tag{44}$$

is facet-defining for $B(G, L, D)$.

■

4 A mixed-integer model

In several variants of the survivability model, in addition to installing enough capacity so as to reroute commodities when an edge or node is deleted, one must also "reserve" capacity on any edge, for any commodity. A version of this requirement is considered in [5]. Given integer $n \geq 2$ and **rational** $L > 0$, the polyhedron $F_n(L)$ is described by the system

$$\sum_{j \neq i} f_j \geq L, \ \forall i, \ 1 \leq i \leq n \tag{45}$$

$$f_i - x_i \leq 0 \tag{46}$$

$$f \geq 0, \qquad x \in Z_+^n \tag{47}$$

Here f_i models the "total" flow on element i of a cutset (or, more accurately, the total capacity actually reserved for use on i), and x_i is the "capacity" installed on that element. For simplicity, we have not included an inequality of the form $\sum_j f_j \geq D$ in this formulation.

The polyhedra $F_n(L)$ are substantially more complex than the $P_n(L, D)$. It can be seen that the projection of $F_n(L)$ to the space of the x variables is $P_n(\lceil L \rceil, \lceil L \rceil)$. But we have come across models where the f_i variables have nonzero costs – typically this occurs where there is an additional discrete capacity allocation, but one with small granularity and that can therefore be linearized (but not ignored since the contribution to the total cost may be nontrivial).

In our computational experience, a large number of "trivial" facet-defining inequalities are violated and should be handled: these are obtained by replacing, in one of the constraints (45) some of the f_j with the corresponding x_j, and applying the MIR procedure [19] to the resulting valid inequality. However, next we will describe a higher-rank inequality that plays a more important role. Given a vector w and a set S, we use the notation $w(S) = \sum_{j \in S} w_j$. We denote $E = \{1, \cdots, n\}$.

We want to cut-off fractional points that satisfy all, or almost all, of the inequalities (45) and (46) nearly tightly. Let $I \subset \{1, \cdots, n\}$. Set $p = |I|$, and assume $n > p > 0$. For any $j \in I$ (45) implies

$$f(I \backslash j) + x(E \backslash I) \geq L \tag{48}$$

The sum of these inequalities gives

$$(p-1) f(I) + p x(E \backslash I) \geq pL \tag{49}$$

Similarly, we obtain

$$(n-p)f(I) \;+\; (n-p-1)x(E\backslash I) \;\;\geq\;\; (n-p)L \tag{50}$$

When both (49) and (50) hold with equality we have

$$x(E\backslash I) \;\;=\;\; \frac{n-p}{n-1}L \;=\; x^* \tag{51}$$

almost certainly fractional (consider Figure 1). If we apply the MIR procedure to (49) and (50) directly we will obtain an inequality that cuts off this fractional point but will usually not even define a face. The problem is that (50) can be tightened:

$$f(I) \;\;\geq\;\; L \;\; - x(E\backslash I) + \lceil \frac{x(E\backslash I)}{n-p} \rceil. \tag{52}$$

(The validity of this inequality follows from the fact that $x_j \;\geq\; \lceil \frac{x(E\backslash I)}{n-p} \rceil$ for at least one element $j \in x(E\backslash I)$, and consequently $f(E\backslash(I \cup j)) \;\leq\; x(E\backslash I) - \lceil \frac{x(E\backslash I)}{n-p} \rceil$. Notice also that this inequality is linear only locally).

It turns out that, depending on n, L, and p, two cases arise that give rise to different inequalites. Assuming x^* is fractional, consider the lower envelope for $f(I)$ (in $(x(E\backslash I), f(I))$-space) in the neighborhood of x^*. For $x(E\backslash I) \;\leq\; \lfloor x^* \rfloor$ this lower envelope is defined by (49) which has slope $-p/(p-1) < -1$. For $x(E\backslash I) \;\geq\; \lceil x^* \rceil$ the lower envelope is defined (locally) by (52) which has slope -1 so long as $\lceil \frac{x(E\backslash I)}{n-p} \rceil$ is constant. Thus, the lower envelope for $f(I)$ in the neighborhood of x^* will either be convex, or not, depending on whether the least possible value for $f(I)$ decreases by at least 1 (or not) as $x(E\backslash I)$ goes from $\lfloor x^* \rfloor$ to $\lceil x^* \rceil$. This corresponds to whether

$$\frac{p}{p-1}(L - \lfloor x^* \rfloor) \;-\; (L - \lceil x^* \rceil + \lceil \frac{\lceil x^* \rceil}{n-p} \rceil) \;\;\geq\;\; 1, \tag{53}$$

or not. Rather than present a (tedious) general analysis, we provide an example for the first case. We write $\hat{x} \;=\; \lfloor x^* \rfloor$ and $\hat{f} \;=\; \frac{p}{p-1}(L - \hat{x})$.

Example 4.1 *Suppose* $L \;=\; 40.6$, $n = 15$, $p = 3$. *Then* $x^* = 34.8$, $\hat{x} = 34$, *and* $\hat{f} = 9.9$. *Say* $I = \{1, 2, 3\}$. *The point* (x^0, f^0) *with*

$$\begin{aligned} x_j^0 \;&=\; 3,\;\; 4 \leq j \leq 14,\;\; x_{15}^0 \;=\; 1 \\ f_j^0 \;&=\; x_j^0,\;\; 4 \leq j \leq 15, \\ f_j^0 \;&=\; 3.3,\;\; 1 \leq j \leq 3, \\ x_j^0 \;&=\; 4,\;\; 1 \leq j \leq 3, \end{aligned} \tag{54}$$

is feasible, and satisfies $x^0(E\backslash I) \;=\; \hat{x}$ *and* $f^0(I) \;=\; 9.9 \;=\; \hat{f}$. *Notice this point satisfies (49) with equality. Also, the point* (x^1, f^1) *with*

$$\begin{aligned} x_j^1 \;&=\; 3,\;\; 4 \leq j \leq 14,\;\; x_{15}^1 \;=\; 2 \\ f_j^1 \;&=\; x_j^1,\;\; 4 \leq j \leq 15, \\ f_j^1 \;&=\; 3,\;\; 1 \leq j \leq 2,\;\; f_3^1 \;=\; 2.6, \\ x_j^1 \;&=\; 3,\;\; 1 \leq j \leq 3, \end{aligned} \tag{55}$$

is feasible, and $x^1(E\backslash I) = 35$ *satisfies (52) with equality, since* $f^1(I) = 40.6 - 35 + \lceil \frac{35}{12} \rceil = 8.6$. *Setting*

$$\alpha \;\;=\;\; \frac{f^0(I) \;-\; f^1(I)}{x^1(E\backslash I) - x^0(E\backslash I)} \;\;=\; 1.3$$

we obtain the inequality

$$f(I) + 1.3x(E\backslash I) \;\; \geq \;\; \hat{f} + 1.3\hat{x} \; = \; 54.1. \tag{56}$$

Inequality (56) is valid, since it is tight when $x(E\backslash I) = 35$ *or* $x(E\backslash I) = 34$ *(and hence it is strictly valid when* $x(E\backslash I) < 34$*) and it is strictly valid when* $x(E\backslash I) = 36$*.*

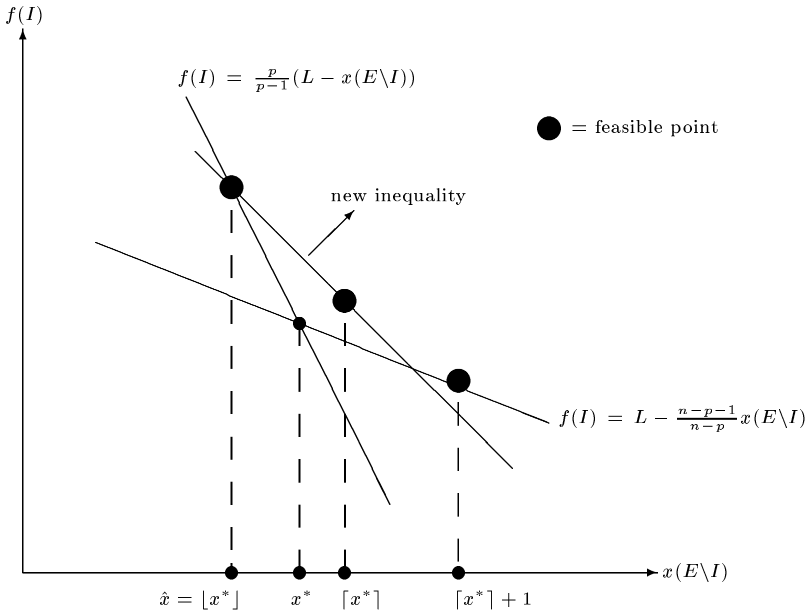

Figure 1: Inequality involving flows

However, this inequality is not facet-defining because all supporting points satisfy $x(E\backslash I) = f(E\backslash I)$*. Instead, we have to "lift" it to an inequality of the form*

$$f(I) + 1.3x(E\backslash I) \;\; \geq \;\; 54.1 + \lambda(x(E\backslash I) - f(E\backslash I)), \tag{57}$$

for appropriate $\lambda \geq 0$*. A calculation yields* $\lambda = \frac{10}{11}$*.*

Refer to [8] for a more complete analysis. From a computational viewpoint, having chosen $|I|$ one computes the parameters α and λ; and then finding a set I that maximizes violation of the inequality is straightforward.

The structure of $F_n(L)$ is fairly complex. One way to cut-off fractional points is to seek inequalities that are violated by points (x, f) with $f(I)$ "too small" for appropriate I. The above is an example of this. A general procedure to obtain strong inequalities is to study the behavior of $f(I)$ as a function of $x(\bar{I})$ (where $\bar{I}$ is another subset), essentially resulting in disjunctive inequalities (i.e. they are tight for $x(\bar{I}) = k$ or $x(\bar{I}) = k + 1$ for some k). These inequalities typically are supported by "efficient" feasible points. Writing $\bar{L} = \lfloor L \rfloor$, these are points with $x_i = \lceil \frac{\bar{L}}{n-1} \rceil$ or $x_i = \lfloor \frac{\bar{L}}{n-1} \rfloor$ and $f_i = \frac{L}{n-1}$ or close to it, for every i. Refer to [8] for several classes of inequalities that exploit these ideas.

In the following section we also describe a facet for a more general problem, a simplication of which is facet-defining for $F_n(L)$.

5 A multicommodity model

The problem we study here can be described as follows: we are given a tri-partite graph with color classes A, M, B (so all edges are of the form $\{a,m\}$ or $\{m,b\}$ with $a \in A$, $m \in M$ and $b \in B$) and a number of demands to be routed from A to B using length-two paths (i.e. paths that use one node in M). Capacities are installed in M and the combined routing/capacity allocation has to be survivable.

More precisely, we are given positive rationals L^k, $k = 1, \cdots, K$, and subsets M^k, $k = 1, \cdots, K$, of set M, and our system of constraints is

$$\sum_{j \in M^k - i} f_j^k \geq L^k, \ \forall i \in M^k \ \forall k, \ 1 \leq k \leq K \tag{58}$$

$$\sum_{k : j \in M^k} f_j^k - x_j \leq 0 \tag{59}$$

$$x \in Z_+^n \qquad f \geq 0 \tag{60}$$

Here f_j^k is the amount of capacity reserved for commodity k on $j \in M^k$, and x_j is the capacity installed in $j \in M$. Constraint (58) states that the routing is survivable and (59) is a capacity constraint. This problem is a simplification of a real problem as described later.

Let P denote the polyhedron given by the above constraints. The structure of P can be complex, and one can derive variants of the inequalities described in previous sections. Next we will instead describe the simplest example of a different kind of inequality. As in the previous section, these are inequalities that are supported by "efficient" points, that is, feasible points that simultaneously have minimal (or near-minimal) capacity and flow allocations. These inequalities are useful in cutting-off fractional points that route too little flow.

Denote $m^k = |M^k|$ for each $k \in K$. For simplicity we assume $m^k \geq 3$ for all k. To motivate the discussion suppose we consider some $k \in K$, and some $j \in M^k$. Then $\sum_{i \in M^k - j} f_i^k$ is minimized by setting $f_i^k = \frac{L^k}{m^k - 1}$ for all $i \in M^k$ (and the remaining variables appropriately). Suppose next that we insist that f_j^k can be at most $\frac{L^k}{m^k-1} - \delta$, for a certain $\delta > 0$ and small. In that case, applying inequalities (58) to all $i \in M^k - j$ yields that $\sum_{i \in M^k - j} f_i^k$ must be increased by $\frac{m^k-1}{m^k-2}\delta$, that is, the value of the program

$$Min \sum_{i \in M^k - j} f_i^k$$

s.t.

$$(f, x) \in P$$

$$f_j^k \leq \frac{L^k}{m^k - 1} - \delta$$

is

$$L^k + \frac{m^k - 1}{m^k - 2}\delta. \tag{61}$$

To extend this idea to multiple commodities, fix an element $j \in M$, and let $K^j = \{k \in K : j \in M^k\}$. Let $H_j = \sum_{k \in K^j} \frac{L^k}{m^k - 1}$, and for an integer $T > 0$, let V^T denote the value of the program

$$Min \sum_{k \in K^j} \sum_{i \in M^k - j} f_i^k$$

s.t.

$$(f,x) \in P$$

$$x_j \leq T$$

We note that the integrality restriction on the x variables can be removed without changing the value of the problem.

Trivially

$$V^{\lceil H_j \rceil} = \sum_{k \in K^j} L^k \tag{62}$$

which is attained by the "ideal" flow $f_i^k = \frac{L^k}{m^k - 1}$ for all i and $k \in K^j$.

Suppose now that $H_j \notin Z$ and we want to compute $V^{\lfloor H_j \rfloor}$. Then the decrease $F_j = H_j - \lfloor H_j \rfloor$ has to be apportioned among the commodities in K^j. Without loss of generality, assume these are commodities $1, \cdots, v$ and $m^1 \geq m^2 \cdots \geq m^v$. As per our observation above (c.f. (61)) we want to apportion as much of the decrease F_j as possible to commodity 1, any leftover amount to commodity 2, and so on. More formally, let t be the smallest index such that

$$\frac{L^1}{m^1 - 1} + \frac{L^2}{m^2 - 1} + \cdots + \frac{L^t}{m^t - 1} \geq F_j. \tag{63}$$

and set

$$\gamma = F_j - \frac{L^1}{m^1 - 1} - \frac{L^2}{m^2 - 1} - \cdots - \frac{L^{t-1}}{m^{t-1} - 1}. \tag{64}$$

Then we have

$$V^{\lfloor H_j \rfloor} = V^{\lceil H_j \rceil} + \sum_{1}^{t-1} \frac{L^i}{m^i - 2} + \frac{m^t - 1}{m^t - 2}\gamma. \tag{65}$$

Let $q = V^{\lfloor H_j \rfloor} - V^{\lceil H_j \rceil}$. We want to show that the inequality

$$q x_j + \sum_{k \in K^j} \sum_{i \in M^k - j} f_i^k \geq q \lceil H_j \rceil + \sum_{k \in K^j} L^k \tag{66}$$

is valid. In other words, we want to show that for any integer $T > 0$, $qT + V^T \geq q\lceil H_j \rceil + \sum_{k \in K^j} L^k$. Clearly this is true when $T \geq \lceil H_j \rceil$ or when $T = \lfloor H_j \rfloor$, by construction. But it can be shown that for any T, $V^{T-1} - V^T \geq V^T - V^{T+1}$, as required.

Proposition 5.1 *[8] Inequality (66) defines a facet of P whenever $H_j \notin Z$.* ∎

The above inequalities can be generalized as follows. Rather than considering a single variable x_j, instead one can use a set. One then obtains facet-defining inequalities involving several variables x_j with different coefficients (see [8]).

5.1 On-going work

We are now engaged in implementing separation routines for some of the inequalities described above. The underlying network design problems are basic "network loading" models as studied in [7], which we will outline here, and simplification of models studied in [5]. The initial computational results are fairly positive and this work will be described in a future paper.

For the network loading case, the inequalities we are using are all of those in Section 2.1 and MIR-like inequalities of the form described in Section 4. For the more complicated models presented in [5] we are using inequalities (3) and inequality (66) (and others not presented in this paper).

In the basic network loading model, we have to assign integral capacities to the edges of a graph so that multicommodity demands can be routed (and we pay for the capacities). The survivability feature further states that if an edge is deleted, enough capacity should exist in the remaining graph so that at least some fraction of the commodities can be routed. There are several variations within this theme, but by looking at a cut in the graph, one obtains systems, or relaxations thereof, of the form studied in Sections 2, 4 and 5. Having selected a cut, we can then separate the inequalities we described above – in the case of the inequalities in Sections 2.1 and 4 a greedy algorithm will find the most violated inequality.

To select a "good" cut, the basic separation technique we follow an approach similar to one used in [7]. The key idea is to find a "tight cut" in the network. Here "tight" means that the capacity of edges in the cut is as close as possible to the total flow across the cut, in the current relaxation. In order to find such a cut, we assign to each edge a weight that grows proportional to the ratio of capacity to flow on that edge, plus a small random perturbation. Then, for every vertex s, we construct its shortest path tree, and for each integer k, $1 \leq k \leq N/2$ (where N is the number of nodes in the graph) we consider the set of k nodes that are closest to s using the current metric (the random perturbations ensure that no ties occur). These neighborhoods are our candidates for "good" cuts.

It has been observed by other authors that adding network survivability features to network design models yields problems where the optimal cost significantly increases (i.e., over the same model without survivability constraints) and that cutting-plane approaches using standard inequalities fail to correspondingly tighten the formulation. In our testing as outlined above we have observed large improvements of the lower bound in a single iteration of separate-and-cut. On the other hand, the linear programs to be solved can become quite difficult, especially for large models. We will describe these results in a forthcoming paper.

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